L.B.9
G...

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

EDITED BY

LESTER R. FORD, Editor-in-Chief

WITH THE CO-OPERATION OF

E. F. BECKENBACH
E. T. BELL
L. M. BLUMENTHAL
N. B. CONKWRIGHT
H. S. M. COXETER
OTTO DUNKEL
H. P. EVANS
HOWARD EVES
B. F. FINKEL
J. S. FRAME
ORRIN FRINK, JR.
MARJORIE GROVES
B. W. JONES
C. V. NEWSOM
HAIM REINGOLD
MARIE J. WEISS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME 53

1946

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND CHICAGO, ILL.

R v.53 1946

Tech

THE AMERICAN MATHEMATICAL MONTHLY.

MONTEREY, VIRGINIA, DEC. 26, 1893.

DEAR SIR:—

As there is no regular Mathematical Journal devoted to the solutions of problems and the discussion of mathematical subjects published in the United States, we have taken upon ourselves the great responsibility of publishing The American Mathematical Monthly, the first number of which will appear in January, 1894.

In this journal, we shall endeavor to publish all problems sent to us by contributors, whether they be the easy problems in Arithmetic or the difficult problems in the Calculus, Mechanics, or Probability, and we earnestly invite every lover of mathematics to send us problems and solutions, and their contributions will receive the proper recognition. We, therefore, earnestly solicit your aid in this undertaking in the way of subscriptions and contributions. The journal is to be issued promptly each month, each issue containing 20 pages or more, 10x7 1-2 inches. The paper is to be of good quality and the typography will receive close attention.

We have already secured the aid of the most eminent Mathematicians in the United States. We name a few: Prof. W. E. Byerly, Professor of Mathematics in Harvard College; Prof. W. S. Hoover, Professor of Mathematics in the Ohio University; Prof. Henry Heaton, Atlantic City, Iowa—a superior Mathematician; Prof. G. B. Zerr, Professor of Mathematics, Staunton Military Academy; Prof. G. B. Halsted, A. M., Ph. D., ex-fellow of Princeton College, and Johns Hopkins University, Professor of Mathematics, University of Texas; etc. These are sufficient to prove to you that the Journal will be of a very high order. No pains will be spared on the part of the Editors to make this the most interesting and practical Mathematical Journal published in America. The subscription price is $2.00 per year. If you will favor us with your subscription, please sign your name to the attached portion of this letter and return to us. Hoping to receive a favorable reply from you by return mail, we remain

Yours very truly,

B. F. Finkel, A. M.,
Professor of Mathematics in Kidder (Mo.) Institute, and member of the N. Y. Mathematical Society.

J. M. Colaw, A. M.,
Principal of High School, Monterey, Virginia, and member of the N. Y. Mathematical Society.

Chubbuck Bros., Publishers, Kidder, Mo.

.....

MATHEMATICS AND LOGIC

A brief survey serving as preface to a review of
The Philosophy of Bertrand Russell

HERMANN WEYL, Institute for Advanced Study

1. Reduction of mathematics to set theory: the logical apparatus. The reduction of mathematics to set-theory was the achievement of the epoch of Dedekind, Frege and Cantor, roughly between 1870 and 1895. As to the basic notion of set (to which that of function is essentially equivalent) there are two conflicting views: a set is considered either a collection of things (Cantor), or synonymous with a property (attribute, predicate) of things. In the latter case "x is a member of the set γ," in formula $x \epsilon \gamma$, means nothing but that x has the property γ. The property of being red or being odd is certainly prior to the set of all red bodies or of all odd numbers. On the other hand, if with regard to a bag of potatoes or a curve drawn by pencil on paper, the property of a potato to be in the bag or of a point to lie on the curve is introduced, then the set (or a more concrete structure representing the set) is prior to the property. Whatever the epistemological significance of this distinction, it leaves the mathematician cool, since for any property γ we may speak of the set γ of all elements which have the property, and with respect to a given set γ we may speak of the property to be a member of γ. When he adopts the term *set* in preference to *property* the mathematician indicates his intention to consider co-extensive properties as identical, two properties α and β being co-extensive if every element that has the property α has also the property β and *vice versa* (set = "Begriffsumfang," Frege). Thus he will identify red and round in spite of their different "meanings" if every red body in the world happens to be round and *vice versa*.

The property π of "being prime" is represented by the propositional function $P(x)$, read "x is prime," with an argument x the range of significance of which is circumscribed by the concept "number." (The natural numbers 1, 2, 3, $\cdots$ shall simply be called numbers; other numbers will be specified as rational, real, *etc.*) Indeed, understanding of the (false) proposition "6 is prime" requires that one understand what it means for *any* number x to be prime. Hence the proposition $P(6)$ arises from the propositional function $P(x)$ by the substitution $x=6$. Besides properties we must consider binary, ternary, $\cdots$, relations, represented by propositional functions of 2, 3, $\cdots$ arguments.

Although the mathematician need care little whether the language of properties or sets is used, he cannot afford to ignore another distinction sometimes confused with it: the distinction between what is considered as *given* and what he *constructs* from the given by the iterated combination of certain explicitly described constructive processes. For instance, in the axiomatic setup of elementary geometry one considers as given three categories of objects—points, lines (=straight lines) and planes, and a few relations between these objects (as "point *lies on* plane"). More complicated relations must be "defined," *i.e.*, logically constructed from these primitive relations. In that intuitive theory of natu-

ral numbers (arithmetic) which is truly basic for all mathematics, even the objects are not given but constructed from the first number 1 by iterating one process, addition of 1; while all arithmetical relations are logically constructed from the one basic relation thus established: $y=x+1$, "x is followed by y." On the other hand, in a phenomenology of nature one would have to deal not only with categories of objects, as "bodies" or "events," but also with whole categories of properties which are prior to all construction, *e.g.* with the continuum of color qualities.

Logical construction of propositional functions from other propositional functions consists in the combined iterated application of a few elementary operations. Among them are the primitive logical operators $\sim$ (not), $\cap$ (and), $\cup$ (or), and the two quantifiers $(\exists x)$, "there exists an x such that," and $(\forall x)$, "for all x." For instance, from two propositional functions $S(x)$, $T(x)$ we form

$$\sim S(x), \qquad S(x) \cap T(x), \qquad S(x) \cup T(x), \qquad (\exists x)S(x), \qquad (\forall x)S(x).$$

The quantifiers carry an argument x as index and "kill" that argument in the propositional function following the quantifier, just as the substitution of an individual number, say $x=6$, does. The arithmetic operations $+$ and $\times$ are primarily applied to numbers and from there carry over to functions, while the process of integration with respect to a variable x by its very nature refers to a function $f(x)$. Just so, the logical operations $\sim$, $\cap$, $\cup$ deal primarily with propositions, while $(\exists x)$, $(\forall x)$ refer to propositional functions involving a variable x. The operator $\cup$ is primitive in the sense that the truth value (true or false) of $a \cup b$ depends only on the truth values of a and b. The same holds for $\sim$ and $\cap$. It is convenient to add the primitive operator of implication for which I use Hilbert's symbol $\rightarrow$: $a \rightarrow b$ is false if a is true and b false, but true for the three other combinations: a true, b true; a false, b true; a false, b false. Propositions without argument result when all variables have been eliminated by substitution of explicitly given individuals or by quantifiers. The construction of arithmetic propositions and propositional functions constitutes their "meanings."

In introducing properties of numbers we presuppose that we know what we mean by "*any* number"; we shall say that the *category* of the elements to which the argument in the propositional function under consideration refers must be given. We are assuming that this category is a closed realm of things existing in themselves, or, as we shall briefly say, is *existential*, when we ask with respect to a given property γ of its elements whether there *exists* an element of the property γ, with the expectation that, whatever the property γ, the question has a definite meaning and that there either exists such an element or every one of its elements has the opposite property $\sim\gamma$. In number theory or in elementary geometry we assume the numbers or the points, lines, planes, to constitute existential categories in this sense. However, we envisage only single individual properties and relations, never anything like the category of "all possible properties of numbers." This situation changes radically with the set-theoretic approach.

There we are forced to consider properties of numbers x as *objects* ξ of a new type to which the numbers stand in the relation $x \epsilon \xi$. The proposition "6 is prime" is now looked upon as arising from the binary relation $x \epsilon \xi$ by substituting 6 for x, and the property π of being prime for the argument ξ. The "copula" ϵ corresponds to the word "is" in the spoken sentence "6 is prime." Any propositional function $P(x)$, like "x is prime," gives rise to a corresponding property $\pi = [x]P(x)$ (the property of being prime) such that $P(x)$ is equivalent to $x \epsilon \pi$. The operator $[x]$ which effects transition from the propositional function to the corresponding property or set kills the argument x. For the sake of uniformity of notation we shall write henceforth $\epsilon(x; \xi)$ instead of $x \epsilon \xi$. In the same way a binary propositional function $P(x, y)$ defines a relation $\pi = [x, y]P(x, y)$ and $\epsilon(xy; \pi)$ expresses the same as $P(x, y)$, namely that x and y are in this relation π.

2. Two examples. Let our further reflections be guided by two typical examples taken from Dedekind's set-theoretic analysis of the two decisive steps in the building up of mathematics: his analysis of the sequence of numbers (in "*Was sind und was sollen die Zahlen*" 1887) and of the continuum of real numbers (in "*Stetigkeit und Irrationalzahlen*" 1872). Our critical attention will be kept more alert by using Frege's terminology of properties rather than that of sets.

I. A property α of numbers is said to be hereditary if for any number x that has the property α, the follower $x+1$ also has it. Dedekind defines: A number b is less than a if there *exists* a hereditary property that a has but b has not.

Here it is not only supposed that we know what we mean by *any* property, but we refer to the totality of all possible properties. In applying the quantifiers to properties of numbers as well as to numbers, it is absolutely imperative to look upon properties as secondary objects related to our primary objects, the numbers, by the copula relation ϵ. Heredity is even a property of properties. To be consistent we have to imagine objects of type 1 (the numbers), of type 2 (the properties of numbers), type 3, $\cdots$, and the fundamental relation $\epsilon(x_n; x_{n+1})$ connects a variable x_n of type n with one x_{n+1} of type $n+1$. Let $\mathrm{I}(\xi)$ denote the proposition that ξ is hereditary. The definition of this propositional function whose Greek argument refers to the category "properties (or sets) of numbers" is as follows:

$$\mathrm{I}(\xi) = (\forall x)\{\epsilon(x; \xi) \rightarrow \epsilon(x + 1; \xi)\},$$

and Dedekind's definition of $x < y$:

$$(x < y) = (\exists \xi)\{\mathrm{I}(\xi) \cap \epsilon(y; \xi) \cap \sim \epsilon(x; \xi)\}.$$

II. Dedekind, as Eudoxos had done more than 2000 years before him, characterizes a non-negative real number α by the set of all positive rational numbers (fractions) $x > \alpha$. But for him any arbitrarily constructed (non-empty) set α of fractions (satisfying a certain condition, namely that along with any fraction b it contains every fraction $> b$) creates a corresponding real number α. The real number α is but a *façon de parler* for this set α. A set α consists of all fractions x satisfying a certain propositional function $A(x)$; $\alpha = [x]A(x)$. Let $\mathrm{I}(\xi)$ be a

propositional function whose argument ξ refers to properties of fractions. The (greatest) lower bound $\gamma = [x]C(x)$ of a set of non-negative real numbers ξ can then be obtained as the *join* of all sets ξ for which $\mathbf{I}(\xi)$ holds:

$$C(x) = (\exists\xi)\{\mathbf{I}(\xi) \cap (x \epsilon \xi)\}. \tag{1}$$

In this way Dedekind *proves* that any set of non-negative real numbers has a lower bound. Again the quantifier $(\exists\xi)$ is applied to "all possible properties of fractions."

3. Levels or no levels? The constructive and the axiomatic standpoints. But now let us pause to think what we have been doing. Properties of fractions are constructed by combined iterated application of a number of elementary logical operations $O_1, O_2, \cdots, O_h$. Let us call any property α thus obtained a *constructible* property, or a property of level 1. We can then interpret $(\exists\xi)$ in the definition (1) as "There exists a *constructible* property ξ," and this application of the quantifier is legitimate provided we admit its applicability to natural numbers. Indeed, the different manners in which one can form finite sequences with iteration out of h symbols $O_1, \cdots, O_h$ is not essentially more complicated than the possible finite sequences of one symbol 1. But the property $\gamma = [x]C(x)$ defined by (1) is certainly not identical in its meaning with any of the properties of level 1 because it is defined *in terms of the totality of all properties of level* 1. It is therefore a property of higher level 2. Nevertheless it may be *coextensive* with a property of level 1, and this is exactly what Russell's "axiom of reducibility" claims. But if the properties are constructed there is no room for an axiom here; it is a question which ought to be decided on the ground of the construction; and in our case that is a hopeless business. On the other hand, the edifice of our classical analysis collapses if we have to admit different levels of real numbers such that a real number is of level $l+1$ when it is defined in terms of the totality of real numbers of level l. If we wish to save Dedekind's proof we must abandon the constructive standpoint and assume that there is *given*, independently of all construction, an existential category "properties" or "second-type objects" (among which the constructible ones form but a small part) such that the following axiom holds, replacing the definition (1): For a given third-type object $\mathfrak{i}$ there exists a second-type object γ such that

$$x \epsilon \gamma \cdot \equiv \cdot (\exists\xi)\{(x \epsilon \xi) \cap (\xi \epsilon \mathfrak{i})\}.$$

Here the arguments x and ξ refer to objects of first and second types, and $\equiv$ means "coextensive." That is a bold, an almost fantastic axiom; there is little justification for it in the real world in which we live, and none at all in the evidence on which our mind bases its constructions. With the assumption that properties constitute an existential category of given objects we return from Dedekind, who wanted to *construct* the real numbers out of the rational ones, to Eudoxos, for whom they were *given* by the points on a line; and instead of proving the existence of the lower bound on the ground of a definition of real numbers, we accept it as an axiom.

In reflecting on the source of the antinomies which had shown up at the fringe of Cantor's general set-theory, Russell realized the necessity of distinguishing between the several levels [1]. No doubt, by this fundamental insight, which he expressed somewhat loosely by his vicious circle principle: "No totality can contain members defined in terms of itself," he cured the disease but, as shown by the Dedekind example, also imperiled the very life of the patient. Classical analysis, the mathematics of real variables as we know it and as it is applied in geometry and physics, has simply no use for a continuum of real numbers of different levels. With his axiom of reducibility Russell therefore abandoned the road of logical analysis and turned from the constructive to the existential-axiomatic standpoint,—a complete volte-face.* After thus abolishing the several levels of properties, he still has the hierarchy of types: primary objects, their properties, properties of their properties, *etc.* And he finds that this alone will stop the known antinomies. But in the resulting system mathematics is no longer founded on logic, but on a sort of logician's paradise, a universe endowed with an "ultimate furniture" of rather complex structure and governed by quite a number of sweeping axioms of closure. The motives are clear, but belief in this transcendental world taxes the strength of our faith hardly less than the doctrines of the early Fathers of the Church or of the scholastic philosophers of the Middle Ages.

4. The Russell universe. Let us describe this structure in a little more detail. A few primary categories of elements are given; they are the ranges of significance for the lowest types of arguments. In a relation, each of the n ($=1$ or 2 or 3 or $\cdots$) arguments $x_1, \cdots, x_n$ refers to a certain "type" $k_1, \cdots, k_n$; the relation itself is of a type $k^* = \{k_1, \cdots, k_n\}$, determined by $k_1, \cdots, k_n$, that stands higher than any of the constituents $k_1, \cdots, k_n$. Draw a diagram representing k^* by a dot and $k_1, \cdots, k_n$ by a row of dots below k^* joined with it by straight lines, as you would depict a man k^* and his descendants in a geneological tree. Descending from k^* to its n constituents, and from them to their constituents, *etc.*, one obtains a "topological tree" in which each *end point* is associated with one of the primary categories: this diagram describes the type k^*. The fundamental relation is $\epsilon(x_1 \cdots x_n; x^*)$ where $x_1, \cdots, x_n$ refer to given types $k_1, \cdots, k_n$ and x^* to $k^* = \{k_1, \cdots, k_n\}$. Existential categories of elements are supposed to be given, one for each possible type (including the lowest, the primary types). We mentioned at the beginning with what data axiomatic elementary geometry operates, or a phenomenology of nature may have to operate; our present "Russell universe" U is seen to be incomparably richer.

* I know very well that this is at odds with Russell's own interpretation; he in the course of time became more and more inclined to visualize sets as "logical fictions." "Though," as Gödel adds, "perhaps the word fiction need not necessarily mean that these things do not exist, but only that we have no direct perceptions of them." In the second edition of Volume I of the Principia Mathematica, an attempt is made to prove independently of the axiom of reducibility that at least all levels of *natural numbers* can be reduced to the five lowest. But as Gödel observes, the proof is far from being conclusive. [2]

We give a few of the more obvious axioms on which its theory is to be erected. The universal normal form for propositional functions involving n variables $x_1, \cdots, x_n$ of given types $k_1, \cdots, k_n$ is $\epsilon(x_1 \cdots x_n; a^*)$. Indeed, such a relation is itself an element a^* of type $k^* = \{k_1, \cdots, k_n\}$. The relation of identity $(x=y)$ between elements of type k is itself an element of type $\{k, k\}$; call it $I = I_k$. Similarly, let $E = E_{k_1 \cdots k_n}$ be the ϵ relation with its $n+1$ arguments of types $k_1, \cdots, k_n, k^* = \{k_1, \cdots, k_n\}$. The existence of these special elements must be stipulated explicitly:

Axiom 1. There is an element $I = I_k$ of type $\{k, k\}$ such that $\epsilon(xy; I)$ holds if and only if the elements x, y of type k are identical. There is an element $E = E_{k_1 \cdots k_n}$ of type $K = \{k_1, \cdots, k_n, k^*\}$ such that $\epsilon(x_1 \cdots x_n x^*; E)$ is coextensive with $\epsilon(x_1 \cdots x_n; x^*)$, the variables $x_1, \cdots, x_n$ and x^* ranging over their respective categories $k_1, \cdots, k_n$ and $k^* = \{k_1, \cdots, k_n\}$.

The composite property "red or round" is no longer constructed from the descriptive properties "red," "round," but belongs with them to the category of properties given prior to all construction. Its existence must be guaranteed by one of the simpler axioms of closure.

Axiom 2. Given an element a^* and an element b^* of type $k^* = \{k_1, \cdots, k_n\}$ there exists an element c^* of the same type such that $\epsilon(x_1 \cdots x_n; c^*)$ is coextensive with

$$\epsilon(x_1 \cdots x_n; a^*) \cup \epsilon(x_1 \cdots x_n; b^*)$$

(each x_i varying over its category k_i).

Substitution of a definite element b for a variable is taken care of by the

Axiom 3. Given an element a^* of type $k^* = \{k_1, \cdots, k_n\}$ and an element b_n of type k_n, there exists an element c of type $k = \{k_1, \cdots, k_{n-1}\}$ such that $\epsilon(x_1 \cdots x_{n-1}; c)$ is coextensive with $\epsilon(x_1 \cdots x_{n-1} b_n; a^*)$ (if $x_1, \cdots, x_{n-1}$ vary over the categories $k_1, \cdots, k_{n-1}$).

Elimination of a variable x_n by the corresponding quantifier $(\exists x_n)$ changes a relation a^* of type k^* into a relation a of type k:

Axiom 4. Given an element a^* of type $k^* = \{k_1, \cdots, k_n\}$, there exists an element a of type $k = \{k_1, \cdots, k_{n-1}\}$ such that $\epsilon(x_1 \cdots x_{n-1}; a)$ is coextensive with $(\exists x_n)\epsilon(x_1 \cdots x_{n-1} x_n; a^*)$.

These are only a few typical axioms that indicate the direction. The reader must not look for them in the *Principia Mathematica*, which are conceived in a different style. But our system U embodies the same ideas in a form that seems to me both natural in itself and advantageous for a comparison with other systems W, Z presently to be discussed.

Our axioms serve as a basis for deduction in the same way as, for instance, the axioms of geometry; deduction takes place by that sort of logic on which one is used to rely in geometry or analysis, including the free use of "there exists" and "all" with reference to any fixed type in the hierarchy of types and the elements of the corresponding categories. While these categories, as well as the basic relation ϵ, are considered as undefined, the logical terms like "not" $\sim$,

"if then" $\rightarrow$, "there is" $(\exists x)$, *etc.*, have to be understood in their meaning and do not form part of the axiomatic system: the formalism of symbolic logic is merely used for the sake of conciseness.

If the *Principia Mathematica* set out to base mathematics on pure logic the result, as we now see, is quite different: an axiomatic world system has taken the place of logic. Its very structure, the hierarchy of types, cannot be described without resort to the intuitive concept of iteration. To develop, in Dedekind-Frege fashion, a theory of natural numbers from this system is therefore an enterprise of doubtful value.

5. A constructive compromise. Realizing the highly transcendental character of the axiomatic universe from which this system deduces mathematics, one wonders whether it is not possible to stick, in spite of everything, to the constructive standpoint, which seems so much more natural to the mathematician. We accept the hierarchy of types; but we assume only one category of primary objects, the numbers; and one basic binary relation between numbers, namely "x is followed by y." All other relations of the various types are explicitly constructed, the quantifiers $(\exists x)$ and $(\forall x)$ being applied only to numbers and not to arguments of higher type. No axioms are postulated. What we can get in this way constitutes the ground level, or level 1. One could build over it a second level containing relations which are constructed by applying the quantifiers to the totality of relations of this or that type constructible on the first level, and proceed in the same manner from the second to a third level, *etc.* One would obtain a "ramified hierarchy" of types and levels. But in this way, as we have said before, nothing resembling our classical Calculus will result. The temptation to pass beyond the first level of construction must be resisted; instead, one should try to make the range of constructible relations as wide as possible *by enlarging the stock of basic operations*. It is *a priori* clear that *iteration* in some form must find a place among these irreducible principles of construction—contrary to the Dedekind-Frege program.

We begin again at the beginning. Let $R(x, \xi)$ be a binary propositional function the two arguments x, ξ of which are of types k, κ respectively; for instance the relation "x is less than ξ" between numbers. We can then speak of the property of a number "to be less than ξ" (or of the set of all numbers less than ξ). This is obviously a property depending on ξ. In general terms we can form $[x]R(x, \xi) = r^*(\xi)$, which is an element of $\{k\}$ depending on ξ such that $R(x, \xi)$ is coextensive with $\epsilon(x; r^*(\xi))$. If in addition to $R(x, \xi)$ we have a propositional function $S(x^*)$ whose argument is of type $k^* = \{k\}$, we can form $T(\xi) = S(r^*(\xi))$, or more explicitly,

$$T(\xi) = S([x]R(x, \xi)).$$

This is the *process of substitution* generating $T(\xi)$ from $R(x, \xi)$ and $S(x^*)$.

Take now the particular situation that $\kappa = \{k\}$. Then argument and value of the function $r^*(\xi) = [x]R(x, \xi)$ are of the same type κ, and whenever that happens *iteration* becomes possible. Thus we define by complete induction a

relation $T(n, \xi)$ in which the argument n refers to the primary category of numbers, as follows:

$$T(1, \xi) = S(\xi);$$
$$T(n + 1, \xi) = T(n, r^*(\xi)) \qquad (n = 1, 2, \cdots).$$

Adding the operation of substitution and iteration, as illustrated by these examples, to the other elementary logic operations, but without applying the quantifiers to anything else than numbers, the writer was able (in *Das Kontinuum*, 1918) to build up in a purely constructive way, and without axioms, a fair part of classical analysis, including for instance Cauchy's criterion of convergence for infinite sequences of real numbers.* In this system iteration plays the role which in set-theory was played by the uninhibited application of quantifiers. Our construction honestly draws the consequences of Russell's logical insight into the tower of levels, which Dedekind had ignored inadvertently and Russell himself, afraid of its consequences, razed to the ground by his axiom of reducibility. Considering their common origin the axiomatic system U as outlined in section 4 and this constructive approach are remarkably different. But even here we have adhered to the belief that "there is" and "all" make sense when applied to natural numbers: in addition to logic we rely on this existential creed and the idea of iteration.

6. Brouwer's intuitionistic mathematics. Essentially more radical and a further step toward pure constructivism is Brouwer's intuitionistic mathematics [3]. Brouwer made it clear, as I think beyond any doubt, that there is no evidence supporting the belief in the existential character of the totality of all natural numbers, and hence the principle of excluded middle in the form "Either there is a number of the given property γ, or all numbers have the property $\sim\gamma$" is without foundation. The first part of the sentence is an *abstract* from some statement of fact in the form: The number thus and thus constructed has the property γ. The second part is one of *hypothetic* generality, asserting something only if $\cdots$; *viz.*, *if* you are actually given a number, you may be sure that it has the property $\sim\gamma$. The sequence of numbers which grows beyond any stage already reached by passing to the next number, is a manifold of possibilities open towards infinity; it remains forever in the status of creation, but is not a closed realm of things existing in themselves. That we blindly converted one into the other is the true source of our difficulties, including the antinomies—a source of more fundamental nature than Russell's vicious circle principle indicated. Brouwer opened our eyes and made us see how far classical mathematics, nourished by a belief in the "absolute" that transcends all human possibilities of realization, goes beyond such statements as can claim real meaning and truth founded on evidence. According to his view and reading of history, classical logic was abstracted from the mathematics of finite sets and

* But of course the theorem of the lower bound of an arbitrary set of non-negative real numbers could not be upheld.

their subsets. (The word finite is here to be taken in the precise sense that the members of such a set are explicitly exhibited one by one.) Forgetful of this limited origin, one afterwards mistook that logic for something above and prior to all mathematics, and finally applied it, without justification, to the mathematics of infinite sets. This is the Fall and original sin of set-theory, for which it is justly punished by the antinomies. Not that such contradictions showed up is surprising, but that they showed up at such a late stage of the game!

Thanks to the notion of "Wahlfolge," that is a sequence *in statu nascendi* in which one number after the other is *freely chosen* rather than determined by law, Brouwer's treatment of real variables is in the closest harmony with the intuitive nature of the continuum; this is one of the most attractive features of his theory. But on the whole, Brouwer's mathematics is less simple and much more limited in power than our familiar "existential" mathematics. It is for this reason that the vast majority of mathematicians hesitate to go along with his radical reform.

7. The Zermelo brand of axiomatics; sets and classes. From this excursion to the left wing of the "constructionists" we return to the universe U with its hierarchy of types. Once one has committed oneself to the existential or axiomatic viewpoint, can one not go forth in the same direction and even erase all differences of types, taking only such precautions as are absolutely necessary to avoid the known contradictions? This is what Zermelo did in his *Untersuchungen über die Grundlagen der Mengenlehre*, 1908 [4]. His axioms deal with only one (existential) category of objects called elements or sets, and one basic relation $x \epsilon y$, "x is member of y." But he is forced to give up the principle that any well-defined property γ determines an element c such that $x \epsilon c$ whenever the element x has the property γ and *vice versa*. Properties are used by him merely to cut out subsets *from a given set*. Hence his axiom of selection: "Given a well-defined property γ and an element a, there is an element a' such that $x \epsilon a'$ if and only if x is member of a and has at the same time the property γ." The notion of a well-defined property which enters into it is somewhat vague. But we know that we can make it precise by constructing properties by iterated combined application of some elementary constructive processes. Instead of saying that x has the property γ, let us say that x is a member of the class γ, $x \epsilon \gamma$. We thus distinguish between elements or sets on the one hand, classes on the other, and formulate the axioms in terms of two undefined categories of objects, elements and classes. Since we postulate that two elements a, b are identical in case $x \epsilon a$ and $x \epsilon b$ are coextensive, and since each element a is associated with the class α of all elements x satisfying the condition $x \epsilon a$, we are justified in identifying a with that class α. Then *every element is a class* and the axioms deal with one undefined fundamental relation $x \epsilon \xi$, "the element x is member of the class ξ," which has absorbed Zermelo's relation $x \epsilon y$ between elements. The principles for the construction of properties are replaced by corresponding axioms for classes; *e.g.*, given two classes α and β, there exists a class γ

such that the statement $(x \epsilon \alpha) \cup (x \epsilon \beta)$ about an arbitrary element x is coextensive with $x \epsilon \gamma$.

Since the axiom of selection can only generate smaller sets out of a given set, we need some vehicle that carries us in the opposite direction. Therefore two axioms are added guaranteeing the existence of the set of all subsets of a given set and the join of a given set of sets. It is essential that they be limited to sets = elements and do not apply to classes.

With the introduction of classes, which is due to Fraenkel, von Neumann, Bernays, and others, the axioms assume the same self-sustaining character as, for instance, the axioms of geometry; no longer do such general notions as "any well-defined property" penetrate into the axiomatic system from the outside. A complete table of axioms for this system, Z as we shall call it, is to be found on the first pages of Gödel's monograph, *Consistency of the Continuum Hypothesis*, Annals of Mathematics Studies No. 3, Princeton, 1940 [5]. Even before the turn of the century Cantor himself had moved in the same direction by distinguishing "consistent classes" = sets and inconsistent classes [6]. Not the hierarchy of types, but the non-admission among the decent "sets" of such classes as are too "big," averts here the disaster of the familiar antinomies.

One might object to a system like Z on the ground that it does not rest on a real insight into the causes of the antinomies but patches up Cantor's original conception by a minimum of concessions necessary to avoid the known contradictions. Indeed, we have no assurance of the consistency of Z, except from the empirical fact that so far no contradictions have resulted from it. But we are in no better position toward Russell's universe U. And Z has its great advantages over U: it is of an essentially simpler structure and seems to be the most adequate basis for what is actually done in present-day mathematics. In particular, the "existential" Dedekind-Frege theory of numbers can be derived from it (Zermelo), and Gödel was able to show (*l.c.*) that Zermelo's far-reaching axiom of choice in a very sharp form is consistent with the other axioms of Z.

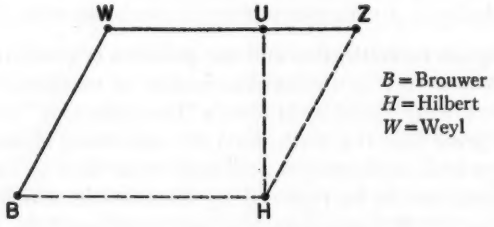
8. Complete formalization and the question of consistency. Pessimistic conclusions. A new turn in the axiomatization of mathematics of paramount importance was inaugurated by Hilbert's "Beweistheorie" (since 1922) [7]. Hilbert sets out *to prove* (not the truth, but) *the consistency of mathematics*. He realizes that to that end mathematics and logic must first be completely formalized: all statements are to be replaced by formulas in which now also the logical operators $\sim$, $\cap$, $(\exists x)$ *etc.*, appear as undefined symbols. Thus formalized logic is absorbed into formalized mathematics.* The formulas have no meaning. A mathematical demonstration is a concrete sequence of formulas in which a formula is derived from the preceding ones according to certain rules comprehensible without recourse to any meaning of the formulas—just as in a game of chess each position is derived from the preceding one by a move obeying certain

* In this regard the ground was well prepared for Hilbert by the Principia Mathematica. For the sake of comparison I mention one other completely formalized system, that of Quine. [8]

rules. *Consistency*, the fact that no such game of deduction may end with the formula $\sim(1=1)$, must be proved by intuitive reasoning about the formulas which rests on evidence rather than on axioms, and respects throughout the limits of evidence as disclosed by Brouwer. But in this thinking about demonstrations, in following a hypothetic sequence of formulas leading up to the end formula $\sim(1=1)$ our mind cannot help using that type of evidence in which the possibility of iteration is founded. In the axiomatization of mathematics Hilbert is forced to proceed with more restraint than Zermelo: if he is too liberal with his axioms he will lose all chance of ever proving their consistency; he is guided by an at least vaguely preconceived plan for such a proof. It is for this reason that he finds it advisable, for instance, to distinguish various levels of variables.

Hilbert's formulas are concrete structures consisting of concrete symbols; the order in which the symbols follow one another in a formula, and also their identity in the same or different formulas, must be recognizable irrespective of little variations in their execution. Handling these symbols, we move on the same level of understanding as guides our daily life in our relationship to such tools as hammer or table or chair. Hilbert sees in it the most important prelogical foundation of mathematics, in fact of all science. But in addition his axioms of mathematics and the intuition of iteration of which the metamathematical non-axiomatic reasoning about mathematics makes use are other extralogical ingredients of his system.

Our brief survey may be summarized in a little diagram in which the constructive tendency increases toward the left, the axiomatic toward the right, and also the relative "depth" of the foundations is indicated. Frege and after him Russell had hoped (1) to develop the theory of natural numbers on a sure basis without recourse to the intuitive idea of indefinite iteration, and (2) to make mathematics a part of logic. We have now seen that none of the systems



discussed gives any hope of accomplishing (2), the subjugation of mathematics by logic. In U and Z elaborate systems of axioms form the basis, in W, B, and to a lesser degree also in U, the intuitive idea of iteration is indispensable. The extra-logical foundations of Hilbert's theory have just been described. The only system which in a sense can claim to reach the goal (1) is Z. But even there the theory of numbers does not rest on logic alone, but on a highly transcendental system of axioms (the belief in whose consistency is supported by empirical facts,

but not by reasons). Poincaré has proved right in his defense of mathematical induction as an indispensable and irreducible tool of mathematical reasoning.

It is likely that all mathematicians ultimately would have accepted Hilbert's approach had he been able to carry it out successfully. The first steps were inspiring and promising. But then Gödel dealt it a terrific blow (1931), from which it has not yet recovered. Gödel enumerated the symbols, formulas, and sequences of formulas in Hilbert's formalism in a certain way, and thus transformed the assertion of consistency into an arithmetic proposition. He could show that this proposition can neither be proved nor disproved within the formalism [9]. This can mean only two things: either the reasoning by which a proof of consistency is given must contain some argument that has no formal counterpart within the system, *i.e.*, we have not succeeded in completely formalizing the procedure of mathematical induction; or hope for a strictly "finitistic" proof of consistency must be given up altogether. When G. Gentzen finally succeeded in proving the consistency of arithmetic [10] he trespassed those limits indeed by claiming as evident a type of reasoning that penetrates into Cantor's "second class of ordinal numbers."

From this history one thing should be clear: we are less certain than ever about the ultimate foundations of (logic and) mathematics. Like everybody and everything in the world today, we have our "crisis." We have had it for nearly fifty years. Outwardly it does not seem to hamper our daily work, and yet I for one confess that it has had a considerable practical influence on my mathematical life: it directed my interests to fields I considered relatively "safe," and has been a constant drain on the enthusiasm and determination with which I pursued my research work. This experience is probably shared by other mathematicians who are not indifferent to what their scientific endeavors mean in the context of man's whole caring and knowing, suffering and creative existence in the world.

REFERENCES

1. Mathematical Logic as Based on the Theory of Type, Am. Jour. Math., v. 30, 1908, pp. 222–262. B. Russell and A. N. Whitehead, Principia Mathematica, 3 vols., Cambridge, 1910–13; 2nd ed. of vol. I, 1935.

2. Gödel, Russell's Mathematical Logic, in The Philosophy of Bertrand Russell, pp. 127, 145, 146.

3. Brouwer's thesis Over de grondslagen der wiskunde appeared in 1907. For a list of his papers on the subject see A. Church's Bibliography of Symbolic Logic, Jour. Symb. Logic, v. 1, 1936, pp. 121–218.

4. Math. Ann., v. 65, 1908, pp. 261–281.

5. See also P. Bernays, Jour. Symb. Logic, v. 2, 1937, pp. 65–77 and the references given there.

6. Cf. G. Cantor, Gesammelte Abhandlungen, ed. E. Zermelo, 1932, pp. 443–451 (correspondence Cantor-Dedekind).

7. Vol. 3 of Hilbert's Collected Papers, 1937, and Hilbert-Bernays, Grundlagen der Mathematik, 2 vols. 1934 and 1939.

8. Am. Math. Monthly, v. 44, 1937, pp. 70–80.

9. Monatsh. Math. Phys., v. 38, 1931, pp. 173–198.

10. Math. Ann., v. 112, 1936, pp. 493–565.

UNIFORM ISOHEDRAL TORI

LEONIDAS ALAOGLU AND J. H. GIESE, Purdue University

1. Introduction. A polyhedron will be said to be *uniform* if the same number of edges radiates from each vertex. It is *isohedral*, in the euclidean sense, if all of its faces are congruent or are mirror images. "Isohedral" is used here in the *local* rather than the *complete* sense, *i.e.* the symmetry groups of the tori considered are not transitive on the faces.

The Euler equation states $F+V-E=2(1-g)$ for a polyhedron of genus g with F faces, V vertices, and E edges. For a uniform isohedral torus (abbreviated u.i.t.) with p-sided faces and e edges per vertex $g=1$, $V=pF/e$, $E=pF/2$, and the Euler equation becomes $F(1-\frac{1}{2}p+p/e)=0$. If $F\neq 0$ the only solutions in positive integers are $p=3$, $e=6$; $p=4=e$; and $p=6$, $e=3$. Thus there may be three classes of u.i.t. with triangular, quadrilateral, or hexagonal faces. In this paper we show the existence of the first two classes by constructing u.i.t. with 1) $p=3$, $e=6$, $F=12n$ $(n\geqq 3)$ or 2) $p=4=e$, $F=8n$ $(n\geqq 4)$. These fragmentary results leave the following questions unanswered. Are there u.i.t. in the hexagonal class? For what values of F can u.i.t. of each class be constructed, and in particular, what is the minimum F in each class? It would also be amusing to know whether there exist knotted u.i.t.

2. Triangular tori. We shall construct triangular u.i.t. by combining suitable octahedral cells. Consider the six points $A(0, 0, h)$, $B(-a, 0, c)$, $C(a, 0, c)$, $D(0, -a, -c)$, $E(0, a, -c)$, and $F(0, 0, -h)$ for a, c, and $h>0$. From the set of segments determined by pairs of these points remove AF, BC, and DE. If $h\neq c$ the remaining segments are edges of an isohedral octahedron. It is easily shown that this is rigid by cutting along edges BD, DC, CE, and EB and hinging edges AB, AC, AD, and AE of one of the halves. For this half DE is a strictly monotone decreasing continuous function of BC. In order to be able to assemble the octahedron we need $BC=DE$. Since both BC and DE can be zero, there is a unique BC such that $BC=DE$. Hence the octahedron is rigid. Since the faces themselves are rigid, excision of CEF and ABD leaves a rigid polyhedron. This implies that *our triangular u.i.t. are rigid.*

Reflect the isohedral octahedron $ABCDEF$ in the plane of CEF to obtain a congruent octahedron $A'B'CD'EF$, where A', B', and D' are the images of A, B, and D respectively. We shall call this pair of apposed octahedra a *wedge*. To prevent interpenetration of octahedra we require them to be convex. Convexity occurs if and only if $h>c$. By symmetry planes BAD, CEF, and $B'A'D'$ intersect in a line. Consider the dihedral angle between these planes which contains our wedge. The dihedral angle between planes BAD and $B'A'D'$ is twice that between planes BAD and CEF. Let the latter angle be α. By taking n congruent wedges with $\alpha=\pi/n$ and apposing face BAD of one wedge to its mirror image $B'A'D'$ in another wedge—in short, by repeated reflections—we

combine the wedges into a torus. Since the octahedra have four edges per vertex and each pair of apposed vertices shares a pair of edges from two octahedra, the torus is uniform with $e=6$. Since each of the $2n$ octahedra shares two faces with other octahedra, the u.i.t. has $12n$ exposed faces.

It remains to determine possible values for n. A simple calculation yields

$$\cos\alpha = (a^2 + 2h^2 - 2c^2)/(a^2 + 2h^2 + 2c^2).$$

Rewrite this as

$$2c^2(1 + \cos\alpha) = (a^2 + 2h^2)(1 - \cos\alpha) \tag{2.1}$$

and recall $0<c<h$ to obtain

$$0 < a^2/h^2 < 2\cos\alpha \csc^2 \tfrac{1}{2}\alpha. \tag{2.2}$$

This implies $\alpha<\frac{1}{3}\pi$. Hence if $\alpha=\pi/n$, $n\geqq 3$. For such an n choose any positive a and h that satisfy (2.2). Then (2.1) determines c. For any $n\geqq 3$ there are two parameters at our disposal. Since their ratio determines the scale or size of our u.i.t. this yields a one parameter family of dissimilar u.i.t. for each $n\geqq 3$.

If one of these u.i.t. has acute angled faces the construction can be modified to permit the insertion of additional faces. First we show that for each $n\geqq 3$ there are u.i.t. with acute-angled faces. Apply the law of cosines to face ACE of our fundamental octahedron to find

$$EA\cdot AC\cos EAC = h^2 - c^2,$$
$$AC\cdot CE\cos ACE = a^2 - 2ch + 2c^2,$$
$$CE\cdot EA\cos CEA = a^2 + 2ch + 2c^2.$$

Since $h>c$, EAC and CEA are acute. ACE will be acute only if $a^2-2ch+2c^2>0$. Use (2.1) to rewrite this as $a^2>2h\cos^2\frac{1}{2}\alpha(c-h\tan^2\frac{1}{2}\alpha)$. Since $h>c$ this will certainly be satisfied if $a^2/h^2>2\cos\alpha$. This lower bound for a^2/h^2 is compatible with the upper bound (2.2) for $\alpha=\pi/n$ for $n\geqq 3$. Hereafter suppose ACE is acute-angled. We now construct an isohedral octahedron different from that previously described. Consider $G(0, 0, z)$, $H(x, 0, 0)$, $I(0, y, 0)$, $J(-x, 0, 0)$, $K(0, -y, 0)$ and $L(0, 0, -z)$ with $xyz\neq 0$. From the set of segments determined by pairs of these points remove GL, HJ, and IK. The remaining segments are the edges of an isohedral octahedron. For suitable x, y, z GHI can be made congruent to any acute angled triangle, in particular to ACE. As before, produce a wedge by reflecting $GHIJKL$ in the plane of JKL. The planes of GHI and their images $G'H'I'$ are parallel. This fact permits the use of this wedge to stretch or spread the u.i.t. previously described.

In the u.i.t. treated heretofore the plane of the common face of any pair of adjacent octahedra will also contain the common face of another different pair of adjacent octahedra. Cut the torus into congruent or enantiomorphic halves by such a plane. If we insert two of our new wedges symmetrically between these halves so that faces GHI are apposed to freshly exposed faces $A'C'E'$ and $G'H'I'$ to ACE we again obtain u.i.t., but with 24 additional faces. By varying

the location of the cut or the number of the new type of wedges symmetrically inserted we can produce u.i.t. with $12n+24k$, $k \geqq 0$, faces congruent or enantiomorphic to ACE. Two u.i.t. with the same n but different k's are dissimilar, so the set of dissimilar u.i.t. with faces congruent to ACE is infinite.

Figure 1 shows a triangular u.i.t. with $n=3$, $k=0$.

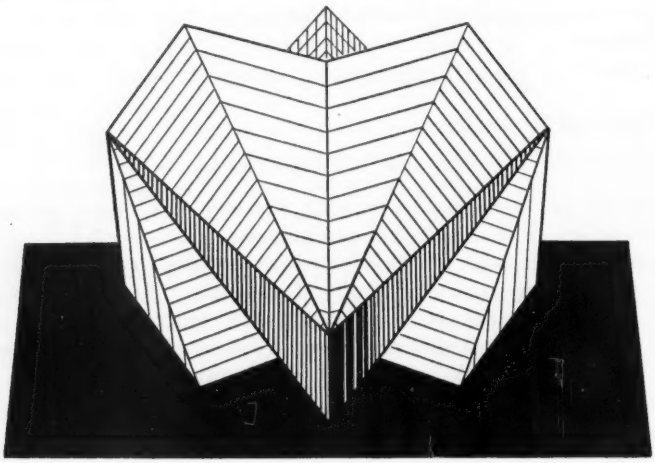


FIG. 1.

3. Quadrilateral tori. We can build $8n$-faced quadrilateral u.i.t. with isohedral hexahedra in a way similar to the first method for constructing triangular u.i.t. with isohedral octahedra. Thus it suffices to construct an isohedral hexahedron, to measure the dihedral angle α between a pair of opposite faces, and to determine the values of n for which we may set $\alpha=\pi/n$.

Consider the points $A(0, 0, h)$, $B(r, 0, c)$, $C(-\frac{1}{2}r, \frac{1}{2}r\sqrt{3}, c)$, $D(-\frac{1}{2}r, -\frac{1}{2}r\sqrt{3}, c)$, $E(r\cos\theta, r\sin\theta, -c)$, $F[r\cos(\theta+\pi/3), r\sin(\theta+\pi/3), -c]$, $G[r\cos(\theta-\pi/3), r\sin(\theta-\pi/3), -c]$, and $H(0, 0, -h)$, where h, r, and $c>0$. It proves helpful to observe that AH is perpendicular to the planes of the congruent equilateral triangles BCD and EFG and passes through their circumcenters. Now we arbitrarily demand that A, B, C, and E be coplanar. This occurs if and only if

$$h = (h - c)[\sin \pi/6 + \sin(\theta + \pi/6)]. \tag{3.1}$$

If (3.1) holds, the following are congruent plane quadrilaterals: $ABCE$, $ACDF$, $ADBG$, $HEFC$, $HFGD$, and $HGEB$. Thus our six points are vertices of an isohedral hexahedron. Let α be the dihedral angle between the planes of the opposite faces $ABCE$ and $HFGD$. By a simple calculation

$$\cos\alpha = [r^2 + 4(h - c)^2 \sin(\theta + \pi/6)]/[r^2 + 4(h - c)^2]. \tag{3.2}$$

Rewrite this as

$$(3.3) \qquad (h - c)^2 = r^2(1 - \cos\alpha)/[\cos\alpha - \sin(\theta + \pi/6)]$$

to find $0<\alpha+\theta<\pi/3$ since $(h-c)^2>0$. Since $0<\theta\leqq\pi/3$, if $\alpha=\pi/n$, then $n\geqq 4$. For such an n we may choose any θ in $0<\theta<\pi/3-\pi/n$ and any r. (3.3) determines $h-c$, and (3.1) determines h. Since r determines the size or scale of the hexahedron, and hence of the u.i.t., the arbitrariness of θ yields a one parameter family of dissimilar u.i.t. for each $n\geqq 4$.

We shall show that the polyhedron that remains after excision of the opposite faces $ABDG$ and $CEFH$ of our isohedral hexahedron is rigid if $\theta\neq\pi/3$. This implies that *our quadrilateral u.i.t. are rigid.* Suppose faces $ABCE$ and $ACFD$ are hinged along AC and opened to some dihedral angle ϕ, $0\leqq\phi\leqq\pi$. The perpendiculars from B and D to AC have a common foot, and if d is the distance from B to AC

$$BD^2 = 2d^2(1 - \cos\phi) \equiv P(\phi).$$

Let e and f be the lengths of the perpendiculars from E and F to AC and g the distance between their feet. Then

$$EF^2 = e^2 + f^2 + g^2 - 2ef\cos\phi \equiv Q(\phi).$$

For faces $BEGH$ and $DFGH$ hinged along GH, by symmetry

$$BD^2 = Q(\phi) \quad \text{and} \quad EF^2 = P(\phi)$$

in terms of the dihedral angle ϕ with edge GH. To permit these four faces to be assembled, there must exist ϕ_1 and ϕ_2 such that

$$(3.4) \qquad P(\phi_1) = Q(\phi_2), \qquad P(\phi_2) = Q(\phi_1).$$

Suppose $P(\phi)-Q(\phi)\equiv 0$. Then for $\phi=0$, $BD=EF=0$, so $BE=CF=CE$, whence $\theta=\pi/3$. By (3.2) $\alpha=0$, which does not interest us. Hence suppose $P(\phi)-Q(\phi)\not\equiv 0$. Since P, Q, and $P-Q$ are monotone continuous functions $\phi_1=\phi_2$. For by (3.4)

$$(3.5) \qquad P(\phi_1) - Q(\phi_1) = -[P(\phi_2) - Q(\phi_2)].$$

By monotonicity $P(\phi)-Q(\phi)=0$ has at most one solution. If $\phi_1\neq\phi_2$, by (3.5) $P(\phi)-Q(\phi)$ actually changes sign so there is a unique ϕ_0 such that $P(\phi_0)-Q(\phi_0)=0$. If $\phi_1<\phi_0$, (3.5) implies $\phi_0<\phi_2$. By (3.4) and monotonicity of P and Q

$$Q(\phi_2) > Q(\phi_1) = P(\phi_2) > P(\phi_1) = Q(\phi_2),$$

a contradiction. $\phi_1>\phi_0$ yields a similar contradiction. Thus the only solution of (3.4) is $\phi_1=\phi_2=\phi_0$, so the hexahedron minus faces $ABDG$ and $CEFH$ is rigid.

Equations (2.1) or (3.1) and (3.3) will provide dimensions of u.i.t. in case it is desired to make cardboard models. Great care must be exercised in cutting out faces to assure that the model will fit together reasonably well. Though it involves extra labor, it will increase the strength of the model to make it out of complete octahedral or hexahedral units as suggested by the construction.

DETROIT PUBLIC LIBRARY

THREE-LINE LATIN RECTANGLES—II

JOHN RIORDAN, Bell Telephone Laboratories

1. Introduction. This note gives a new formula for the numbers, ${}_3K_n$, of reduced three-line Latin rectangles of n elements. It may be recalled from a former paper (this MONTHLY, Oct. 1944, pp. 450–452) that a Latin rectangle is an array in which each row is a permutation of n elements and each column has distinct elements, and that a rectangle is reduced when the elements in the first row are in a standard order.

The formula is as follows:

$$ {}_3K_n = \sum_{i=0}^{[\frac{1}{2}n]} \binom{n}{i} D_i D_{n-i} u_{n-2i},$$

$$D_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)!, \tag{1}$$

$$u_n = \sum_{i=0}^{n} (-1)^i \frac{2n}{2n-i} \binom{2n-i}{i} (n-i)!, \qquad u_0 = 1.$$

Here $D_n = \Delta^n 0!$ is sub-factorial n, a "rencontres number," and u_n is a "menages number," defined as in the former paper (except that there $u_0 = 2$). An equally explicit but more complicated formula has recently been given by Dulmage (this MONTHLY, Oct. 1945, p. 458).

For numerical concreteness, a short table of all numbers in the formula follows:

n	0	1	2	3	4	5	6	7	8
D_n	1	0	1	2	9	44	265	1854	14833
u_n	1	−1	0	1	2	13	80	579	4738
${}_3K_n$				2	24	552	21280	1073760	70299264

Since $D_n < n!$, $u_n < n!$ (by definition of the combinatorial problems they solve), an immediate consequence of the formula is

$$\frac{{}_3K_n - D_n u_n}{n!^2} < \sum_{i=2}^{[\frac{1}{2}n]} \binom{n}{i} i!(n-i)! \frac{(n-2i)!}{n!^2} < \frac{n}{2} \frac{1}{(n-3)^4},$$

which is sufficient to prove the asymptotic formula

$${}_3K_n \sim n!^2 e^{-3}$$

of the former paper.

Further improvements in the inequality above are of course possible but are perhaps of less interest now because of the development of a general asymptotic formula, as yet unpublished, for r-rowed rectangles by Drs. Erdös and Kaplansky, who have kindly sent me a preliminary communication.

2. Proof of Formula (1). By equation (4) of the former paper

$$ {}_3K_n = \sum C_{[a]} u_{[a]}, \tag{2} $$

summation being over all non-unitary partitions of n, each partition denoted by $[a] \equiv [2^{a_2}3^{a_3}, \cdots, n^{a_n}]$, say, with $\sum i a_i = n$, and

$$ C_{[a]} = \frac{n!}{2^{a_2}a_2! \cdots n^{a_n}a_n!}, \tag{3} $$

$$ u_{[a]} = \sum u_{2\pm 2\pm \cdots \pm 2\pm \cdots \pm n\pm \cdots \pm n}. \tag{4} $$

On the right of (4) the subscript contains a_2 2's, a_3 3's, *etc.*, and summation is over the 2^{k-1}, $k = a_2 + a_3 + \cdots + a_n$, possible assignments of $+$ and $-$ signs. Note that $u_{-n} = u_n$. Ex. gr.

$$ \begin{aligned} u_{[532^2]} &= \sum u_{2\pm 2\pm 3\pm 5} \\ &= u_{12} + 2u_8 + u_6 + u_4 + 3u_2. \end{aligned} $$

It is clear that $u_{[a]}$ may be expanded in terms of u_{n-2i}, say

$$ u_{[a]} = \sum_i b_{n,i}^{[a]} u_{n-2i}; \tag{5} $$

and formula (1) will follow if

$$ \sum_{[a]} C_{[a]} b_{n,i}^{[a]} = \binom{n}{i} D_i D_{n-i}, \tag{6} $$

where summation is again over all non-unitary partitions of n (indicated by $[a]$).

For this, the following lemmas are needed.

LEMMA 1: *$b_{n,i}^{[a]}$ is the number of ways of choosing (non-unitary) partitions of i from the parts in $[a]$, repeated parts taken as distinct.*

Proof: The coefficient of u_{n-2i} in the expansion of $u_{[a]}$ is obtained by choosing —signs for all parts, or sets of parts, of $[a]$ whose sum is i. Ex. gr. for $i = 5$ and $[a] = 532^2$, $n = 12$, the coefficient of u_2, as given above, is 3, corresponding to parts 5, 32_1, 32_2, writing 2_1 and 2_2 for the repeated parts. These correspond, respectively, to the following choices of sign:

$$ \begin{gathered} 2 + 2 + 3 - 5 \\ 2 - 2 - 3 + 5 \\ -2 + 2 - 3 + 5 = 2 - 2 + 3 - 5. \end{gathered} $$

LEMMA 2: *The number of ways in which a partition $p_1^{i_1}p_2^{i_2} \cdots$ of i may be chosen from parts $p_1^{j_1}p_2^{j_2} \cdots$ of $[a]$ is*

$$ \binom{j_1}{i_1}\binom{j_2}{i_2} \cdots . $$

Proof: Each repeated part p_k may be chosen in $\binom{j_k}{i_k}$ ways, independently of the others.

Remark: $b_{n,4}^{[a]}$ is the sum of the result of this lemma over all partitions of i contained in $[a]$. *E.g.*, for the example already given,

$$b_{12,4}^{[532^2]} = 1 + \binom{1}{1}\binom{2}{1} = 3.$$

Lemma 3. The contribution of any partition $p_1^{i_1}p_2^{i_2}\cdots$ to the product $b_{m,4}^{[a]}C_{[a]}$ is

$$\binom{n}{i}\frac{i!}{p_1^{i_1}i_1!p_2^{i_2}i_2!\cdots}\,\frac{(n-i)!}{p_1^{j_1-i_1}(j_1-i_1)!p_2^{(j_2-i_2)}(j_2-i_2)!\cdots}.$$

Proof: This follows at once from the identity

$$\binom{j_1}{i_1}\binom{j_2}{i_2}\cdots\frac{p^{i_1}i_1!p^{i_2}i_2!\cdots}{p_1^{j_1}j_1!p_2^{j_2}j_2!\cdots} = \frac{1}{p_1^{j_1-i_1}(j_1-i_1)!p_2^{j_2-i_2}(j_2-i_2)!\cdots}.$$

Equation (6) now follows by summation, noting that

$$\sum_{[a]} C_{[a]} = D_n,$$

which may be regarded as an equation of definition of D_n. Ex. gr. take $i=4$, $n=11$; the partitions of 4 in question are 4 and 2^2, the partitions of 11 may be taken in two classes with coefficients as follows:

Partition:	74	542	4^23	432^2;	72^2	52^3	432^2	32^4
Coefficient:	1	1	2	1	1	3	1	6

Note that 432^2 appears in both classes, since it contains both 4 and 2^2. Then by lemma 3

$$\sum b_{11,4}^{[a]}C_{[a]} = \binom{11}{4}4!\left[\frac{1}{4}+\frac{1}{2^2\cdot 2!}\right]\cdot 7!\left[\frac{1}{7}+\frac{1}{5.2}+\frac{1}{4.3}+\frac{1}{3.2^2.2!}\right]$$
$$= \binom{11}{4}D_4D_7.$$

When $n=2m$, $i=m$, equation (6) fails; the corresponding result is

$$\sum b_{2m,m}^{[a]}C_{[a]} = \binom{n}{m}\frac{1}{2}D_m^2. \tag{7}$$

This exceptional case is avoided in (1) by the choice $u_0=1$.

THE GRAM MATRIX AND HADAMARD THEOREM

C. J. EVERETT AND H. J. RYSER, University of Wisconsin

1. Introduction. The recent book of Halmos [2] on vector spaces has as its object the development of the theory by elegant "coordinate-free" methods which lend themselves to generalization to infinite spaces. We offer here in the same manner proofs of some theorems on the Gram matrix. The complex case is considered, the real case may be treated analogously (and more simply).

2. Inner product spaces. Let $V(C)$ be a *vector space* over the complex field C, *i.e.*, a set of vectors $(0, \xi, \eta, \cdots)$ forming an abelian group under addition, and admitting the complex field $C=(0, 1, a, b, \cdots)$ as operator domain. This means that for every ξ, η of V and every a, b of C, we suppose $(\xi+\eta)a=\xi a+\eta a$; $\xi(a+b)=\xi a+\xi b$; $\xi(ab)=(\xi a)b$; $\xi 1=\xi$.

We say $V(C)$ is an *inner product space* in case for every ξ, η of V exists a complex number (ξ, η) called *inner product*, such that $(\xi+\eta, \zeta)=(\xi, \zeta)+(\eta, \zeta)$; $(\xi, \eta a)=(\xi, \eta)a$; $(\xi, \eta)=\overline{(\eta, \xi)}$ (complex conjugate); $(\xi, \xi)\geqq 0$, and $(\xi, \xi)=0$ if and only if $\xi=0$.

In such a space, vectors ξ, η are called *orthogonal* in case $(\xi, \eta)=0$, and a set of vectors $(\delta_1, \cdots, \delta_n)$ are said to be *orthonormal* in case $(\delta_i, \delta_j)=\delta_{ij}$, the "Kronecker delta." An orthonormal set is of course linearly independent in the sense that $\sum\delta_i a_i=0$ implies that all $a_i=0$.

3. Inequalities of Bessel and Schwarz. Suppose $(\delta_1, \cdots, \delta_m)$ is an orthonormal set in $V(C)$, and ξ is arbitrary. Define $x_i=(\delta_i, \xi)$ as the δ_i-*component* of ξ (Fourier coefficient). We recall the following important relation (R) easily verified by direct computation:

$$(R) \qquad 0 \leqq (\xi - \sum \delta_i x_i, \xi - \sum \delta_i x_i) = (\xi, \xi) - \sum \bar{x}_i x_i \leqq (\xi, \xi).$$

From this follows the *Bessel inequality*

$$(B) \qquad \sum |x_i|^2 = \sum \bar{x}_i x_i \leqq (\xi, \xi)$$

where equality holds if and only if $\xi=\sum\delta_i x_i$.

Note also that the latter inequality of (R) becomes equality if and only if all $x_i=(\delta_i, \xi)=0$, that is, if and only if ξ is orthogonal to all δ_i.

It seems not generally recognized that the Schwarz inequality

$$(S) \qquad |(\xi, \eta)|^2 \leqq (\xi, \xi)(\eta, \eta)$$

is a trivial special case of the Bessel inequality. Most recent books establish (S) independently, and with less elegance.

If $\eta=0$, (S) is trivial. If $\eta\neq 0$, define $\delta=\eta/(\eta, \eta)^{1/2}$ which is then an orthonormal set of *one* element. Using (B) we have for $x=(\delta, \xi)$, $|(\delta, \xi)|^2\leqq(\xi, \xi)$. Substituting for δ and multiplying we get (S). Moreover equality holds in (S) exactly when it does correspondingly in (B), that is when $\xi=\delta(\delta, \xi)$ and ξ, η are dependent.

4. The Schmidt Process. In any treatment of inner product spaces, the Schmidt orthonormalization process plays a fundamental rôle. It may be stated in the following form:

If $(\delta_1, \cdots, \delta_m, \xi_{m+1})C$, where the δ_i are orthonormal, is the subspace generated by the independent vectors δ_i, ξ_{m+1}, then $(\delta_1, \cdots, \delta_m, \xi_{m+1})C = (\delta_1, \cdots, \delta_m, \delta_{m+1})C$, where the latter are *all* orthornormal. Explicitly, one defines $\delta^*_{m+1} = \xi_{m+1} - \sum_1^m \delta_i(\delta_i, \xi_{m+1})$ and then normalizes by choosing $\delta_{m+1} = \delta^*_{m+1}/(\delta^*_{m+1}, \delta^*_{m+1})^{1/2}$.

Thus, if $(\xi_1, \cdots, \xi_n)$ is any set of independent vectors, one obtains for the space $W = (\xi_1, \cdots, \xi_n)C$ an orthonormal basis $(\delta_1, \cdots, \delta_n)$ so that $W = (\delta_1, \cdots, \delta_n)C$, by first defining $\delta_1 = \xi_1/(\xi_1, \xi_1)^{1/2}$ and then inductively producing bases $(\delta_1, \xi_2, \cdots, \xi_n)$, $(\delta_1, \delta_2, \xi_3, \cdots, \xi_n)$, $\cdots$, $(\delta_1, \cdots, \delta_n)$ for W by means of the definition $(\delta_1, \cdots, \delta_m, \xi_{m+1}, \xi_{m+2}, \cdots, \xi_n) = (\delta_1, \cdots, \delta_m, \delta_{m+1}, \xi_{m+2}, \cdots, \xi_n)$ T_{m+1}, $m = 0, 1, \cdots, n-1$. One verifies readily that for the process indicated,

$$T_{m+1} = \begin{bmatrix} 1 & \cdots & & x_1 & & 0 \\ & \ddots & & \vdots & & \\ & & 1 & x_m & & \\ & & & d_{m+1} & & \\ & & & & 1 & \\ & & & & & \ddots \\ 0 & & & & & 1 \end{bmatrix}$$

where $x_i = (\delta_i, \xi_{m+1})$, and $d_1 = (\xi_1, \xi_1)^{1/2}$, $d_{m+1} = (\delta^*_{m+1}, \delta^*_{m+1})^{1/2} \leqq (\xi_{m+1}, \xi_{m+1})^{1/2}$. The latter inequality is due to (R) of §3, and as indicated there, *is equality if and only if ξ_{m+1} is orthogonal to $\delta_1, \cdots, \delta_m$, the $x_i = (\delta_i, \xi_{m+1})$ being zero in T_{m+1}.*

Successive substitution shows that the final basis $(\delta_1, \cdots, \delta_n)$ is related to the original by $(\xi_1, \cdots, \xi_n) = (\delta_1, \cdots, \delta_n)T$, where $T = T_n T_{n-1} \cdots T_1$. It is then clear* that $|T|^2 = d_1^2 \cdots d_n^2 \leqq (\xi_1, \xi_1) \cdots (\xi_n, \xi_n)$.

We show that the equality holds here if and only if the ξ_i are mutually orthogonal. Indeed, if equality holds, *every* $d_i^2 = (\delta_i^*, \delta_i^*) = (\xi_i, \xi_i)$. But $(\delta_2^*, \delta_2^*) = (\xi_2, \xi_2)$ implies, by the italicized remark, that ξ_2 is orthogonal to $\delta_1 = \xi_1/(\xi_1, \xi_1)^{1/2}$, hence to ξ_1. Moreover, δ_2^* and δ_2 are then defined as scalar multiples of ξ_2. An obvious induction completes the proof in this direction.

Now supposing that all ξ_i are mutually orthogonal, we have ξ_2 orthogonal to $\delta_1 = \xi_1/(\xi_1, \xi_1)^{1/2}$, hence by the italicized remark, $(\delta_2^*, \delta_2^*) = (\xi_2, \xi_2)$. Moreover, δ_2 is then defined as a scalar multiple of ξ_2, and ξ_3 is orthogonal to δ_1 and δ_2. Hence $(\delta_3^*, \delta_3^*) = (\xi_3, \xi_3)$, and so on. Finally $|T|^2 = (\xi_1, \xi_1) \cdots (\xi_n, \xi_n)$.

Thus we have

---

* The notation $|T|$ denotes the determinant of the matrix T.

THEOREM 1. *If $\xi_1, \cdots, \xi_n$ are independent, then the Schmidt process gives $(\xi_1, \cdots, \xi_n)C = (\delta_1, \cdots, \delta_n)C$ where $(\xi_1, \cdots, \xi_n) = (\delta_1, \cdots, \delta_n)T$, with $|T|^2 \leqq (\xi_1, \xi_1) \cdots (\xi_n, \xi_n)$, equality holding if and only if the ξ_i are mutually orthogonal.*

5. The Gram matrix. For vectors $\xi_1, \cdots, \xi_n$ in $V(C)$ we define the Gram matrix $G(\xi) = ((\xi_i, \xi_j))$, and prove

THEOREM 2. *The rank of the Gram matrix $G(\xi)$ is equal to the maximum number of linearly independent vectors in the set $\xi_1, \cdots, \xi_n$. If the ξ_i are independent, $G(\xi) = \overline{T}^T T$, where T is the Schmidt matrix of §4†. Thus $|G(\xi)| = |T|^2 > 0$.*

Note first that if ξ_i, η_i, $i = 1, \cdots, n$ are any vectors for which $\xi_i = \sum \eta_j t_{ji}$, we have $(\xi_i, \xi_k) = \sum \bar{t}_{ji}(\eta_j, \eta_l)t_{lk}$ so that for the Gram matrices, $G(\xi) = \overline{T}^T G(\eta) T$.

Now the rank of $G(\xi)$ is unchanged under permutations of rows and columns, so that we may suppose the ξ_i numbered so that the first r are independent, while the remaining ξ_i are expressible in terms of them. Applying the Schmidt process to the first r ξ's enables us to write $(\xi_1, \cdots, \xi_r, \xi_{r+1}, \cdots, \xi_n) = (\delta_1, \cdots, \delta_r, \xi_{r+1}, \cdots, \xi_n)T$ where

$$T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & 0 \end{bmatrix}$$

and T_{11} is the Schmidt matrix. Thus $G(\xi) = \overline{T}^T G(\delta_1, \cdots, \delta_r, \xi_{r+1}, \cdots, \xi_n)T$ where the latter G has in the upper left corner the rxr identity I_r.

Since the rank of the product of matrices is not greater than that of any factor [3, p. 218], rank $G(\xi) \leqq r$. But the upper left corner of $G(\xi)$ is the nonsingular $\overline{T}_{11}^T T_{11}$. Thus rank $G(\xi) \geqq r$. Hence $|G(\xi)| = 0$ unless the ξ are independent. In the latter case, $G(\xi) = \overline{T}^T I T = \overline{T}^T T$ and $|G(\xi)| = |T|^2 > 0$.

6. The Hadamard theorem. *We now obtain trivially* [1]

THEOREM 3. (*Hadamard-Fischer*) *If $\xi_1, \cdots, \xi_n$ are vectors in $V(C)$ then $|G(\xi)| \leqq (\xi_1, \xi_1) \cdots (\xi_n, \xi_n)$, equality holding if and only if the ξ are mutually orthogonal.*

If the ξ are dependent $|G(\xi)| = 0$ and the theorem is trivial. If independent, $G(\xi) = \overline{T}^T T$ and the result follows from THEOREM 1.

BIBLIOGRAPHY

1. E. Fischer, Über den Hadamardschen Determinantensatz, Archiv f. Math., 13, 1908, 32–40
2. P. Halmos, Finite dimensional vector spaces, Annals of Math. Studies No. 7, 1942, Princeton U. Press.
3. C. C. MacDuffee, An Introduction to Abstract Algebra, Wiley, New York, 1940.

† If $T = (t_{ij})$ then T^T means the matrix (t_{ji}), *i.e.*, the "transposed" matrix with rows and columns interchanged.

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans 18, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

BINOMIAL IDENTITIES

R. E. GREENWOOD AND A. M. GLEASON, U.S.N.R., Washington, D.C.

The use of probability concepts to establish algebraic identities is not new. The problems of Chapters II and III of Uspensky's *Introduction to Mathematical Probability* furnish several examples of such use. This article will be concerned with certain non-trivial identities for binomial coefficients which were discovered by the use of probability considerations.

The authors were led to this problem when they noted that computed values of Q and R, as defined below, were equal for fixed N, n, and m, where $0 \leqq n \leqq N$, $0 \leqq m \leqq N$. The defining relations for Q and R are

$$Q = \sum_p \binom{N-n}{p}\binom{n}{m-p}\binom{N-p}{n}, \tag{1}$$

$$\max(0, m-n) \leqq p \leqq \min(m, N-n),$$

and

$$R = \sum_q \binom{n}{q}\binom{N-n}{m-q}\binom{N-q}{N-n}, \tag{2}$$

$$\max(0, m+n-N) \leqq q \leqq \min(n, m).$$

If one actually tabulates the binomial coefficients above, certain similarities in Q and R may be observed. This similarity is brought into focus by making the substitution $q=(m-p)$ in (2). One then obtains

$$R = \sum_p \binom{n}{m-p}\binom{N-n}{p}\binom{N-m+p}{N-n}, \tag{3}$$

$$\max(0, m-n) \leqq p \leqq \min(n, N-n),$$

in which the similarity to (1) is more marked. Attempts to find a direct proof of the identity $Q \equiv R$ by manipulations with factorials were unsuccessful. A proof was devised, however, in which a double summation form would reduce to Q when one of the indicated summations had been carried out, and would reduce to R if the other indicated summation had been carried out. The method of this proof was used to establish a more general set of binomial relations.

Let there be three boys, Tom, Dick, and Harry, and let them make independent choices of k, m, and n integers respectively from the set of the first N

positive integers. What is the probability that there is no integer common to the three choices? Or, since the use of probabilities introduces denominator terms, what is the total number of ways in which the three choices can be made so that there is no common integer?

Suppose that Tom's set of k integers and Dick's set of m integers contain p duplications. If Harry's set of n integers does not duplicate any of these p integers, a "favorable" case has been obtained. Hence one way of counting the number of favorable cases is to introduce the parameter p, count the number of favorable cases for each allowable value of p, and perform a summation on these terms. Let Tom select his k integers from the entire collection of N. This can be done in $\binom{N}{k}$ ways. For fixed p, Dick's choice of m integers may be regarded as the sum of two choices: p of Dick's selections are to coincide with selections already made by Tom, and the remaining $(m-p)$ of Dick's selections must be chosen from the $(N-k)$ integers not selected by Tom. Thus Dick has $\binom{k}{p}\binom{N-k}{m-p}$ ways of selecting his m integers so that there are exactly p duplications between his selection and Tom's selection. To insure no integer common to all three selections, Harry need only select his n integers from the $(N-p)$ integers which are not included in both Tom's and Dick's selections. Harry, therefore, has $\binom{N-p}{n}$ ways of making a favorable selection. Altogether there are

$$\binom{N}{k}\binom{k}{p}\binom{N-k}{m-p}\binom{N-p}{n}$$

ways in which Tom and Dick have exactly p duplications none of which is chosen by Harry. All favorable cases are obtained by summing on all allowable values of the parameter p. The following inequalities limit p,

$$\max(0, m+k-N) \leqq p \leqq \min(k, m, N-n).$$

The number of favorable cases, P, is given by

$$P = \sum_p \binom{N}{k}\binom{k}{p}\binom{N-k}{m-p}\binom{N-p}{n} \tag{4}$$

with the range on p given above.

A similar expression for P can be obtained by interchanging Tom's and Dick's roles. One obtains

$$P = \sum_p \binom{N}{m}\binom{m}{p}\binom{N-m}{k-p}\binom{N-p}{n}. \tag{5}$$

Both (4) and (5) reduce to

$$(6) \qquad P = \sum_{p} \binom{N-p}{n} \frac{N!}{p!(k-p)!(m-p)!(N-k-m+p)!}.$$

Thus a direct proof of the symmetric nature of P with respect to the variables k and m is evident from (6).

In general, it will not be necessary to specify the range of the summation variables, since the binomial coefficients may be assigned the value zero whenever negative factorials are involved.

There are four other expressions for the number P similar to (4) and (5). These four expressions are obtainable from a count of the number of favorable cases when other groupings of Tom, Dick, and Harry (or other permutations of k, m, and n) are taken. Since P is the same number in all these cases, it follows that the form (5) is symmetric in the three variables k, m, and n. Since a direct proof of the symmetry of form (5) for all three variables does not follow from simple algebraic manipulations of the factorials, form (5) may be termed a non-trivial symmetric form.

To exhibit the full symmetry of form (5) in the three variables k, m, and n, another method of calculating the number P is preferable. Let Tom and Dick choose p integers in common, Dick and Harry choose q integers in common, and Harry and Tom choose r integers in common. Then Tom has $(k-p-r)$ choices left, Dick has $(m-p-q)$, and Harry has $(n-q-r)$. There are $(N-k-m-n+p+q+r)$ integers not chosen at all. The number of ways of choosing these seven categories can be summed over all allowable values for p, q, and r, and one obtains

$$(7) \quad P = \sum_{pqr} \frac{N!}{p!q!r!(k-p-r)!(m-p-q)!(n-q-r)!(N-k-m-n+p+q+r)!}$$

for the number of favorable cases. Bearing in mind that p, q, and r are dummy summation parameters, the symmetry in k, m, and n is evident in the above formulation.

Form (7) can be summed on one of the summation parameters to give other expressions for P. Thus

$$(8) \qquad \begin{aligned} &\sum_{r} \frac{1}{r!(k-p-r)!(n-q-r)!(N-k-m-n+p+q+r)!} \\ &= \sum_{r} \binom{k-p}{r} \binom{N-m-k+p}{n-q-r} \frac{1}{(k-p)!(N-m-k+p)!} \\ &= \binom{N-m}{n-q} \frac{1}{(k-p)!(N-m-k+p)}. \end{aligned}$$

The substitution of (8) in (7) yields

$$(9) \quad P = \sum_{pq} \frac{N!(N-m)!}{p!q!(m-p-q)!(n-q)!(k-p)!(N-m-n+q)!(N-m-k+p)!}.$$

Form (9) (or the two forms symmetric thereto) can also be obtained by counting up the number of favorable cases in another way. For example, Tom's set of k integers and Dick's set of m integers may have p integers in common while Dick and Harry may have chosen q in common which do not duplicate any of the set of p integers. These choices may be counted up as before, and the resulting expression for P will reduce to form (9) when factorials are used.

If form (7) be summed out on two of the parameters, using identities similar to (8), a form similar to (4), (5) or (6) is obtained. Thus a direct proof of the symmetrical nature of form (5) can be obtained independently of probability considerations by showing that the various forms may all be derived from (7) by algebraic operations.

To return to the first non-trivial identity, $Q \equiv R$, note that if $k = (N-n)$, and if $\sum_r \sum_q$ be performed on (7), $\binom{N}{n} Q$ is obtained, and that if $\sum_r \sum_p$ be performed on (7), $\binom{N}{n} R$ is obtained. Thus a direct proof of the identity follows by performing two sets of transforming operations on (7).

A THEOREM CONCERNING CIRCLES*

VICTOR THÉBAULT, Tennie, Sarthe, France

The note published by W. M. Stewart under the above title (this MONTHLY, 1937, p. 165) is editorially accompanied by the following theorem of F. E. Wood:

If three circles are drawn having as centers the vertices of a triangle and radii ρk, ρl, ρm, respectively, where k, l, m are fixed, then as ρ varies, the locus of the radical center of the three circles is a straight line (or part of a line if only real circles are used).

This proposition is implicitly contained in a study by J. Neuberg (*Mathesis*, 1923, p. 298) in connection with a problem concerning some circles proposed by E. Lemoine (*Assoc. Franc. pour Avanc. des Sciences*, 1888, p. 170). It was also pointed out by A. Dechilly (*Journal de Vuibert*, t. 36, 1912, p. 91) with an application to the case where k, l, m are equal to the opposite sides of the triangle.

On our part, we have extended the problem to the tetrahedron and have obtained the following theorem which led us, in particular, to the determination of a *sphere of Longchamps* for the tetrahedron analogous to the circle of the same name for a triangle (*Mathesis*, 1932, pp. 223–229; *L'Enseignement Mathématique*, Genève, 1937, pp. 81–99).

THEOREM. *In a tetrahedron $T \equiv ABCD$, the locus of the center of the sphere (ω), for which the powers of the vertices A, B, C, D have the forms kl^2, km^2, kn^2, kp^2, (l, m, n, p being given values and k being an arbitrary coefficient), as k varies, is a straight line drawn perpendicularly through the circumcenter of T to the plane having barycentric coordinates l^2, m^2, n^2, p^2.*

* Translated from the French by Howard Eves.

This theorem has a number of particular cases of which some have given rise to interesting remarks contained in our aforementioned articles. (*Cf.* also V. Thébault, *Gazeta Matematica*, Bucarest, t. 41, pp. 172–175.)

Recently we have considered four spheres (ω), (ω'), (ω''), (ω''') such that the powers of the vertices A, B, C, D of T, with respect to each of them, are (kl^2, km^2, kn^2, kp^2), (km^2, kn^2, kp^2, kl^2), (kn^2, kp^2, kl^2, km^2), (kp^2, kl^2, km^2, kn^2). Concerning these spheres we have the following

THEOREM. *The sphere orthogonal to the spheres* (ω), (ω'), (ω''), (ω''') *has for center the centroid* G *of* T *and for the square of its radius*

$$\sigma^2 = [4k(l^2 + m^2 + n^2 + p^2) - S]/16,$$

S being the sum of the squares of the edges of T.

If one interchanges the same two powers of the set kl^2, km^2, kn^2, kp^2 in each of the sequences of powers for (ω), (ω'), (ω''), (ω''') one obtains four new spheres (ω_1), (ω_2), (ω_3), (ω_4), which we call spheres *semi-associated* with (ω). Concerning these spheres we have the

THEOREM. *Four spheres semi-associated with* (ω) *have for radical center the centroid G of T. The radical planes of* (ω) *with respect to its semi-associated spheres pass through the medians AG, BG, CG, DG of T. A sphere* (ω) *and all its semi-associated spheres are orthogonal to the same sphere* (G, σ).

Finally, the values $(-kl^2, -km^2, -kn^2, -kp^2)$ are the powers of the vertices A, B, C, D with respect to a sphere (ω', ρ') whose center ω' is the symmetric with respect to the circumcenter O of T of the center ω of the sphere (ω, ρ) corresponding to the powers (kl^2, km^2, kn^2, kp^2), and, if R is the circumradius of T, we have the relation

$$\rho'^2 + \rho^2 = 2(R^2 + d^2),$$

which becomes $\rho'^2 = 2(R^2 + d^2)$ when the sphere (ω, ρ) reduces to its center ω, d being the distance $O\omega$.

If one changes the sign of one of the powers of A, B, C, D to obtain successively $(-kl^2, km^2, kn^2, kp^2)$, $(kl^2, -km^2, kn^2, kp^2)$, $(kl^2, km^2, -kn^2, kp^2)$, $(kl^2, km^2, kn^2, -kp^2)$, these new powers correspond to some spheres (Ω_1), (Ω_2), (Ω_3), (Ω_4). We may state the following theorem concerning these spheres.

THEOREM. *The radical planes of the sphere* (ω, ρ) *with respect to each of the spheres* (Ω_1), (Ω_2), (Ω_3), (Ω_4) *are the planes of the faces of T, and the radical centers of quadruples of spheres obtained by taking the sphere* (ω, ρ) *with triples of the spheres* (Ω_1), (Ω_2), (Ω_3), (Ω_4) *are the vertices of T. The radical center of the spheres* (Ω_1), (Ω_2), (Ω_3), (Ω_4) *has barycentric coordinates* $1/l^2$, $1/m^2$, $1/n^2$, $1/p^2$, *and coincides with the pole with respect to T of the radical plane of the spheres* (O, R) *and* (ω, ρ).

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

CLUB REPORTS 1944–45

Pi Mu Epsilon, Northwestern University

Six regular meetings of the Illinois Beta Chapter were held throughout the year. The topics presented were as follows:

The dynamics of the solar system, by Dr. Oliver Lee

Pi by chance, by Dr. Elliot Buell

Alexandria, the home of mathematics, by Dr. H. T. Davis

Mathematics and electrical engineering courses, by Dr. Robert Beam

Soap films and the problem of Plateau, by Dr. W. T. Reid

Some applications of elementary calculus to economic theory, by Dr. William Jaffe

The geometry of numbers, by Dr. A. T. Lonseth.

A party was held in the fall at the home of the social chairman. The annual initiation banquet was held on May 19, with Dr. Richard Wolfe as the toastmaster. The banquet address was

Mathematical research during the war, by Dr. E. J. Moulton.

Officers for the year were: President, Isabel Barrett; Vice-President, John Pederson; Secretary, Betty McDonald; Treasurer, Paul Axt; Social Chairman, Priscilla Williams; Scholarship Chairman, Julia Bissell; Faculty Adviser, Dr. Elliot Buell; Permanent Secretary, Dr. E. H. C. Hildebrandt.

Mathematics—Physics Club, College of Saint Teresa

The club met monthly and devoted its meetings to the history of mathematics and physics. There were reports on

Mathematicians and physicists: Galileo, Johannson, Richards

Women in science: Herschel, Germain, Agnesi, Noether

Catholic scientists: Galvani, Volta, Ampère, Roentgen

American mathematicians: D. E. Smith, Cajori, Karpinski.

Two plays were given, entitled

Archimedes, and

The cubic equation.

The members were addressed by a guest speaker on

Science and industry.

At the close of the year a picnic was given in the nature of a Mathematical Convention of Historical Characters.

Officers were: President, Josephine Schmelzle; Vice-President, Kathleen Raleigh; Secretary, Sara Horn; Treasurer, Joyce Hassett.

Mathematical Society, University of Adelaide, South Australia

On June 22, 1944, a Mathematical Society was founded with the object of promoting interest in mathematics among undergraduates, graduates, and teachers of mathematics at schools in the metropolitan area. Meetings are held once a fortnight and papers are read dealing either with mathematical topics that have been discussed or mentioned in advanced courses given during the year at the University, or with original mathematical work actually done by the speaker.

Membership is informal and open to all persons who have some fundamental knowledge of mathematics in general and an interest in the development of some branch of Pure or Applied Mathematics. One object of the Society is to enable those who are interested in mathematics to keep in touch with modern developments in the subject, and it is hoped that teachers in secondary schools and students taking post-graduate courses will for that reason become active members.

Following the preliminary meeting on June 22, the following topics were presented during the winter term from June to October 1944.

The theory of cyclic groups, by G. E. Wall

Vector algebra and matrix calculus, by A. T. James

Some fundamental notions of mathematical statistics, by B. Worthley

On approximative methods for the solution of analytic equations, by D. R. Blaskett

The isoperimetric problem—an application of Fourier series, by H. Schwerdtfeger

Steffensen's formula on divided differences, by G. J. Aitchison

Arithmetical functions, by Mrs. M. Sved

Derivation of the equation of the vibrating string from physical principles, by W. O. Gibberd.

Mathematics Club, University of Dayton

Biweekly meetings were held during the year at which members of the club presented the following papers:

Problems in logic, by Frank Levin

Topology, by Theodore Brown

Probability, by Catherine Ens

Mathematics and the ancients, by Kenneth Trimbach

The hatchet planimeter, by George Igel

A mathematical interpretation of a passage from the fourth Georgic of Vergil, by Allan Braun

Transfinite numbers, by Frank Levin

Roman methods of calculating, by Alice Blaeser

Diophantine analysis, by John Thalheimer

Application of graphs to the solution of rate and mixture problems, by Margaret Magin

The Doolittle methods of solving linear equations, by Helen Lechner.

Faculty talks presented in the first and second semesters were:

The summability of a divergent series, by Dr. K. C. Schraut

Strategy in mathematical thinking, by Professor Henry Blumberg of Ohio State University.

During the first semester a new form of social entertainment was introduced into the Club, namely a Monte Casino night. The purpose of this form of entertainment was to illustrate the laws of probability. Toward the close of the second semester a banquet was held at Wishing Well Inn.

At the formal initiation ceremony eighteen new members were issued certificates of membership into the Club. The Vice-President's charge to the new members was entitled

The Mathematics Club—1944, by Allan Braun.

The Dean of Science Award was conferred upon Theodore Brown for having delivered before the Club the most interesting paper of the first semester and upon Allan Braun for the second semester. The book chosen for the Awards was *Isaac Newton* by L. T. More. The Mathematics Club Alumni Award for Excellence in Advanced Mathematics was conferred at the June Commencement upon George Igel of the senior class and upon Alice Blaeser of the junior class.

During the year, the Mathematics Club was encouraged to sponsor mathematics clubs in several high schools in the vicinity of Dayton. The project proved to be so popular that an organizing committee composed of representatives from these high schools and the University of Dayton decided to establish an organization which has come to be known as The National Mathematics Honor Society of Secondary Schools. The Mathematics Club of Purcell High School in Cincinnati and Chaminade High School in Dayton were admitted as the first Chapters of the Society. In May the University of Dayton Mathematics Club sponsored a colloquium at which representatives of the High School Chapters presented papers. Awards for the best papers were presented at a banquet which followed.

The officers of the Club during the year were: President, George Igel; Vice-President and Secretary, Allan Braun; Treasurer, Kenneth Trimbach; Faculty Adviser, Professor K. C. Schraut.

Pi Mu Epsilon, University of Nebraska

This year six meetings were held including business meetings and general meetings. The following talks were presented:

Trisecting the angle, by Miss Mary Kathryn Cooper and Mr. Oscar Ellis

Interpolation, by Mr. Kotaro Murai, Mr. Anton Kashas, and Miss Natalie Burn.

The officers for 1945–46 are: Director, Margaret Stewart; Secretary, Mary Lou Weaver; Treasurer, Mary Ann Mattoon. The Faculty Adviser is Professor F. S. Harper.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Spherical Trigonometry after the Cesàro Method. by J. D. H. Donnay. New York, Interscience Publishers, Inc., 1945. 11+83 pages. **$1.75.**

Among the sundry ills of the teaching of elementary mathematics, there are two which in the reviewer's opinion are serious, widespread, and chronic. One is the occurrence of rote methods including, of course, emphasis on the mere acquisition of manipulative techniques. The other is essentially a frame of mind—a rigid and reactionary orthodoxy that insists on the strict segregation of mathematical concepts into compartments in accordance with well established custom. The first of these need not concern us here, since it is not prevalent at the college level. The second however operates to make freshman courses less *interesting*, *comprehensive*, and *fundamental* than the subject matter warrants. Perhaps there is a tendency to assume of our first year students too much computational persistence and not enough conceptual maturity. Certainly, it is not true that all of the concepts, processes, and methods of higher mathematics are beyond the understanding of lower division students. Various topics from vector analysis, the ausdehnungslehre, and the domain of invariance are admirably suited to the exposition of the theory of plane and spherical triangles and two and three dimensional geometry in general. The advantages are simplicity, conciseness, the introduction of the concepts themselves with the resulting expansion of the student's horizon, the development of mathematical maturity, and finally the enthusiasm engendered by the use of powerful methods. A fresh instance of the elementary application of advanced ideas (here the idea is conformal mapping by stereographic projection) is furnished by the Cesáro method for spherical trigonometry—the subject matter of the book under review.

This work is dedicated to the memory of Giuseppe Cesàro (1849–1939), crystallographer and mineralogist, professor at the University of Liége. It was originally planned as a joint writing of the author and Cesàro. We learn from the preface that the material has been used for some time in Belgium and recently at the Johns Hopkins University and Laval University. It seems to the reviewer that the method presented is far superior to the usual procedure.

The general plan is to reduce spherical trigonometry to its plane counterpart by projecting the given triangle onto a plane. Specifically, one of the three vertices A is singled out, call it the north pole, and the triangle is projected onto the equatorial plane from the south pole as center of projection. The two sides which join at the north pole go into straight line segments and the third side goes into the arc of a circle. Angles are preserved. The chord line is drawn and the resulting plane triangle is taken as the mate of the original. The angles

of the plane triangle are then A, $B-E$, $C-E$—if E denotes one-half of the spherical excess. Surely, this is the simplest way to introduce the excess. The sides of the plane triangle are $\sin a/2$, $\sin b/2 \cos c/2$, $\sin c/2 \cos b/2$. By completing the lune determined by sides a and c of the given triangle a second plane triangle, called the derived triangle, is formed. Its angles and sides are $180-A$, E, $A-E$, $\cos a/2$, $\sin b/2 \sin c/2$, $\cos b/2 \cos c/2$. The same procedure may be applied to the polar triangles if desired.

Turning to the details, the following propositions are proved: Circles are projected as circles or straight lines; The angle between two circles is projected in true magnitude. The methods are synthetic and the tools are well known theorems of plane geometry. A knowledge of solid geometry is not required. Some of the demonstrations are believed to be original. All are in elegant form.

After the various parts of the plane triangles have been evaluated in terms of the elements of the original spherical triangle, one could proceed to solve numerical problems in spherical trigonometry by a direct treatment of the corresponding plane triangles. The author prefers to develop the standard formulas of spherical trigonometry and does this in chapters III and IV by applying ordinary trigonometry to the plane triangles and simplifying the results. Thus he first obtains Napier's proportions by applying the law of tangents to the various triangles. From the law of sines he gets Delambre's formulas. Euler's formula for $\cos E$ is obtained by applying the law of cosines to the derived triangle. The derived triangle yields also Lhuilier's and Cagnoli's formulas and a relationship expressing $\cot E$ in terms of two sides and the included angle.

Chapter IV presents the following material: (1) the equation, $\cos a = \cos b \cos c + \sin b \sin c \cos A$; (2) expressions for the half angles in terms of the sides; (3) the well known relationship between two sides and the angles opposite; (4) the formula $\cos A = -\cos B \cos C + \sin B \sin C \cos a$; (5) the expressions of the half-sides in terms of the angles; and (6) some relations connecting four consecutive parts.

Chapter V is devoted to right triangles. Chapter VI is taken up with numerical computations. The final chapter presents a miscellany of theorems which may be established by means of the formulas of spherical trigonometry. The book contains an adequate index, two appendices (spherical areas and formulae of plane trigonometry) and fifty-four problems most of which have answers.

In conclusion, the reviewer feels that the author has made a substantial contribution in presenting the Cesàro method. The general idea of solving a difficult problem by setting up a correspondence with a simpler problem is exciting in itself, and certainly such notions as conformal mapping, stereographic projection, projection and section are well worth acquiring. Thus if the Cesàro route is selected instead of the usual procedure, the student will not only have a more interesting trip through the subject but he will gain more in mathematical maturity—and mathematical maturity makes up perhaps a major portion of the profit derived from studying mathematics.

H. V. CRAIG

A Course of Mathematical Analysis for Higher Technical Institutes. Second Edition. By A. F. Bermant. Moscow and Leningrad, OGIZ, GOSTEKHIZDAT, 1944. Vol. I, 320 pages+120 Figs. $2.00. Vol. II, 336 pages+90 Figs. $3.60.

During the past few years, higher mathematics has been employed to an unprecedented extent in the course of technical investigations, especially in such domains as aerodynamics, elasticity, plasticity, electronics, electrical network analysis, *etc.*; with the further development of research in these fields, it is not unreasonable to expect a parallel increase in the role which mathematics will play in these investigations. Moreover, the recent development of automatic calculating devices, such as punched-card machines, improved differential analyzers and electrical network computers, offers new and unanticipated possibilities for mathematics to supplement experiment in the construction and elaboration of physical theories. These developments in engineering methods demand corresponding modifications in the curriculum and point of view of the engineering school.

Two stringent requirements must be satisfied by a modern mathematical text-book designed for use in a technical school. On the one hand, the book must make a strong appeal to the student's intuition; the student must always be made aware of the relation between abstract mathematical theory and the geometrical, mechanical and physical problems in which it can be applied; on the other, the student must be equipped to study the more advanced parts of higher mathematics, so that he may be able to acquire the knowledge and techniques he may need in his work, and that he may come to appreciate and understand the abstract approach characteristic of mathematical reasoning today. Consequently, the course must help in developing a mental flexibility and resourcefulness which will allow these engineers who devote themselves to theoretical research to formulate practical problems in abstract mathematical terms and at the end to translate the results of analysis into the language of the practicing engineer.

These problems have been brilliantly solved by A. F. Bermant in his "Course of Mathematical Analysis." Professor Bermant is a well known Russian mathematician, managing editor of the *Matematicheski Sbornik* (*Recueil mathematique*), with many years' experience teaching in technical schools. It is quite evident that this experience has strongly influenced the choice of materials, while the intuitive presentation, expository skill and lucidity of style combine to make this book the basis for an excellent course in mathematics for engineering students. Because the technical student regards mathematics primarily as a tool, and not an end in itself, the author avoids formalism whenever he can, appealing constantly to the student's common sense. No opportunity to give a geometrical or mechanical interpretation of mathematical concepts is overlooked, and an illustrative diagram is usually added. Considerable effort is made to present analysis as coherent and rational. For example, each chapter has a brief introduction which indicates what topics are to be considered, where they fit in the framework

of analysis, how they are related to the student's previous knowledge, and why they are important to the engineer. It was obviously impossible to give rigorous proofs of all assertions in a book of this kind, but all statements and definitions satisfy modern standards of accuracy, and what cannot be proved is at least made intuitively clear, the student being referred to more elaborate treatises for the proofs. It is regrettable that the barrier of language renders this book unavailable for use by American students; it would be desirable if it were to be translated. Some idea of the subjects treated may be obtained from the following abbreviated Table of Contents.

VOLUME I: INTRODUCTION. Chap. I. THE FUNCTION CONCEPT; Chap. II. LIMITS; Chap. III. DERIVATIVES AND DIFFERENTIALS. DIFFERENTIAL CALCULUS; Chap. IV. THE BEHAVIOR OF FUNCTIONS. CURVE TRACING; Chap. V. DEFINITE INTEGRAL; Chap. VI. INDEFINITE INTEGRAL. INTEGRAL CALCULUS; Chap. VII. APPLICATIONS OF THE INTEGRAL; VOLUME II: FUNCTIONS OF SEVERAL VARIABLES: Chap. VIII. DERIVATIVES AND DIFFERENTIALS; Chap. IX. DIFFERENTIAL CALCULUS AND SIMPLE APPLICATIONS; Chap. X. MULTIPLE INTEGRALS AND ITERATED INTEGRALS; Chap. XI. LINE INTEGRALS AND SURFACE INTEGRALS; Chap. XII. DIFFERENTIAL EQUATIONS; Chap XIII. SERIES.

An indication of the author's approach may be obtained from the following detailed contents of two representative chapters, those on Differential Equations and Infinite Series. Chap. XII. DIFFERENTIAL EQUATIONS 1. Equations of the first order (Generalities; Equations whose variables can be separated; Total differential equations; Integrating factor.) 2. Equations of the first order (continuation) (Direction fields; Approximate solutions of first order equations; Singular solutions; Clairaut's equation; Orthogonal and isogonal trajectories). 3. Equations of second and higher order (Generalities; Equations of special types; Linear homogeneous equations; Linear non-homogeneous equations). 4. Linear equations with constant coefficients (Homogeneous equations; Non-homogeneous equations). 5. Vibrations (Mechanical vibrations; Free and forced vibrations) Resonance; Oscillations in electric circuits. 6. Concluding remarks. Linear equations which can be reduced to equations with constant coefficient—Systems of equations. Chap. XIII. SERIES 1. Numerical series (The nature of series; Convergence; Series with positive terms; Sufficient conditions for convergence; Leibniz's criterion; Absolute convergence). 2. Series of functions (Definitions. Uniform convergence; Differentiation and integration of series of functions). 3. Power series (Interval of convergence; Abel's theorem, General properties; Taylor series (Application to the solution of differential equations) Definition of the elementary functions; Euler's formulas). 4. Trigonometric series (Preliminary remarks; Fourier series; Expansion of periodic functions; Points of discontinuity of the first kind; Dirichlet's theorem; Remarks on the expansion of functions in Fourier series; Examples; Practical harmonic analysis; Equation of the vibrating string).

STEFAN BERGMAN

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND HOWARD EVES

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to Howard Eves, College of Puget Sound, Tacoma 6, Washington.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 701. *Proposed by N. S. Mendelsohn, Queen's University*

Suppose the distance between the front and rear wheels of a car is b, and the distance between wheels from side to side is a. The right front wheel is turned through a counterclockwise angle θ from its neutral position. Through what angle must the left front wheel be turned so that in the car's subsequent motion all four wheels will roll without slipping? Describe the motion.

E 702. *Proposed by J. Brenner, Palmer Laboratory, Princeton, N. J.*

The smallest consecutive cubes whose difference is a square are 8^3 and 7^3. What is the next pair?

E 703. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In which scales of notation can a four-digit number $abbc$ be the square of a two digit number mn, if $c=a+1$ and $n=m+1$? (*Cf.* E 652 [1945, 42])

E 704. *Proposed by D. H. Bowne, Buffalo, N. Y.*

Sum the series $1+1/3-1/5-1/7+1/9+\cdots$.

E 705. *Proposed by J. Rosenbaum, Bloomfield, Conn.*

A point in the interior of a simple closed curve, which is nowhere concave, will be called a *power point* if the product of the segments of a variable chord through the point is constant. The existence of how many distinct power points insures that the curve is a circle?

SOLUTIONS

The Generalized Water-fetching Puzzle

E 451 [1941, 65]. *Proposed by W. E. Buker, Pittsburgh Public Schools*

There are three containers, having capacities of a, b, c quarts, where $a>b>c$ (positive integers). With the largest container full and the others empty it is desired to divide the liquid into two equal portions, using these containers and no others. For what values of a, b, c is a solution possible?

Solution by J. C. Currie, Northeast Junior College of L. S. U., Monroe, Louisi-

ana. A set of three integers for which the prescribed division is possible will be referred to as *a solution*.

This discussion will be restricted by the following conditions:

(1) $a<b+c-2$. For discussion of solutions not thus restricted see D. H. Browne's partial solution [1942, 125–27].

(2) b and c have no common factor.

(3) a is not a multiple of c (to eliminate a trivial case).

The discussion is based on the conclusions of the editorial note to Browne's solution; the object is to seek solutions not given by the ENSC (Editorial Note Sufficient Condition). We define

$$\sigma_0 = 0, \qquad \sigma_1 = c,$$

$$\sigma_{i+1} = \sigma_i + c \text{ or } \sigma_i - b \text{ according as } \sigma_i < b \text{ or } \sigma_i \geqq b.$$

The σ's form a cycle of $b+c$ distinct integers, a permutation of $0, 1, \cdots, b+c-1$. A set of integers (a, b, c) is a solution if and only if it is possible to pass from either end of the cycle to $\frac{1}{2}a$ without passing, en route, any number greater than a.

If there exists an integer j such that $\sigma_i \leqq 2\sigma_j < b+c-2$ either for every $i<j$ or for every $i>j$ then $(2\sigma_j, b, c)$ is a solution. Of course we reject the j's for which $2\sigma_j$ does not meet the requirement $a>b$, and condition (3).

We shall refer to this as the σ-test.

Let $b=mc+r$, $0<r<c$. Let $c=qr+t_1$, $0<t_1<r$, where condition (2) implies that r and t_1 have no common factor. Let $t_0=0$, $t_{i+1}=t_i+t_1$ or t_i+t_1-r according as $t_i+t_1<r$ or $t_i+t_1 \geqq r$. The t's thus form a cycle of r distinct integers, a permutation of $0, 1, \cdots, r-1$. The σ-test will be replaced by an equivalent test on the t-cycle, by means of which we may construct all solutions for given r and t_1.

The structure of the σ-cycle is revealed to best advantage in a row and column array; for example, for $b=41$, $c=12$ (with $m=3$, $r=5$, since $41=3\cdot 12+5$):

0	12	24	36	48
	7	19	31	43
2	14	26	38	50
	9	21	33	45
4	16	28	40	52
	11	23	35	47
	6	18	30	42
1	13	25	37	49
	8	20	32	44
3	15	27	39	51
	10	22	34	46
	5	17	29	41

Note that the array has c rows, $m+2$ columns; the horizontal lines divide the array into r sections such that within each section successive rows differ by r. The first row contains only multiples of c; r is the last (and least) number in the second column and b is the last (and least) number in the last column. Finally, the rt's are in the first column, the c next larger numbers are in the second column, *etc.*, the c the largest numbers being in the last column.

THEOREM 1. *If* (a, b, c) *is a solution, with* $b>3c$, *then* $(a-2c, b-2c, c)$ *also is a solution.*

The proof depends on the fact that the σ-array for $b-2c$ and c may be obtained from the σ-array for b and c simply by deleting the last two columns. Since there is a difference of c between elements of successive columns of the array, if the greatest number passed on the way to σ_j in the original array does not exceed $2\sigma_j$, then the greatest number passed on the way to σ_j-c in the reduced array does not exceed $2\sigma_j-2c$ (σ_j in the first row of multiples of c having been excluded). The fact that σ_j-c is actually in the reduced array follows from the fact that $\sigma_j=\frac{1}{2}a$ cannot be in the last column of the original array (because of condition (1)) so that σ_j-c cannot be in either of the two deleted columns.

By induction we obtain

THEOREM 2. *If* (a, b, c) *is a solution, then* $(a-2kc, b-2kc, c)$ *also is a solution for every positive integer* k *for which* $b-2kc>c$.

The largest k for which this is possible gives what we may call a "primitive" solution (a, b, c), in which $b=c+r$ or $b=2c+r$, and from which non-primitive solutions may be constructed by adding $2kc$ to a and to b. Note that no non-primitive solutions satisfy the ENSC, except the trivial cases in which a is a multiple of c. For example, the solution (18, 17, 4) given by William Scott [1944, 592] is non-primitive, its primitive being (10, 9, 4).

Case A. Here $b=c+r$, in which case the array has three columns. Inspection of the array for this case reveals that $\frac{1}{2}a$ can only be one of the elements $c+t_j$ (for each j, the first number in the second column of the $(j+1)$th section), and that the elements $2c+t_i$ (for each i, the first number in the third column of the $(i+1)$th section) are the crucial blocking elements; *i.e.*, passage by these implies passage by other elements. Thus the inequalities of the σ-test may be written in the sharper form $2c+t_i\leqq 2c+2t_j<b+c-2$, implying the following test for case *A*:

Given r and t_1, if there exists an integer $j<r$ such that $t_i\leqq 2t_j<r-2$, either for every $i<j$ or for every $i\geqq j$, then $(2c+2t_j, c+r, c)=(2qr+2t_j+2t_1, (q+1)r+t_1, qr+t_1)$ is a solution for $q=1, 2, \cdots$.

Most of the resulting solutions are of the ENSC type, but for $r\leqq 20$ there are 14 j's which give solutions not of this type. The first one is $r=13$, $t_1=4$, $j=11$, giving solutions $(26q+18, 13q+17, 13q+4)$ for $q=1, 2, \cdots$.

Case B. Here $b=2c+r$, in which case the array has four columns. Inspection of the array for this case reveals that $\sigma_j=\frac{1}{2}a$ can only be in the third column of the array since elements in first and second columns cannot satisfy the left

inequality of the σ-test, and elements of the fourth column cannot satisfy the right inequality. Of the third column elements, those in the rth or last section, $c+nr$, give solutions $(2c+2nr,\ 2c+r,\ c)$ for every positive integer $n \leqq \frac{1}{2}(q+1)$. All these solutions satisfy the ENSC and, with the ones mentioned under case A, exhaust the solutions of the ENSC type.

Since, as stated above, $\frac{1}{2}a$ must be in the third column of the array, for sections other than the rth, $\frac{1}{2}a$ must be of the form $c+nr+t_j$, $1 \leqq n \leqq q+1$. Then $a=2c+2nr+2t_j$. We note that the r largest numbers in the last column, $3c+t_i$, are crucial blocking elements, so that the inequalities of the σ-test may be recast: $3c+t_i \leqq 2c+2nr+2t_j < 3c+r-2$. Recalling that $c=qr+t_1$, this implies $t_i \leqq (2n-q)r+2t_j-t_1 < r-2$, which permits only $n=\frac{1}{2}q$ if q is even and $n=\frac{1}{2}(q+1)$ if q is odd. This leads to two tests for case B.

I. When q is even.

Given r and t_1, if there exists an integer $j<r$ such that $t_i \leqq 2t_j-t_1<r-2$ either for every $i<j$ or for every $i \geqq j$, then $(3qr+2t_j+2t_1,\ (2q+1)r+2t_1,\ qr+t_1)$ is a solution for $q=2, 4, 6, \cdots$.

II. When q is odd.

Given r and t_1, if there exists an integer $j<r$ such that $t_i \leqq r+2t_j-t_1<r-2$ either for every $i<j$ or for every $i \geqq j$, then $((3q+1)r+2t_j+2t_1,\ (2q+1)r+2t_1,\ qr+t_1)$ is a solution for $q=1, 3, 5, \cdots$.

There are, for $r \leqq 20$, 164 examples under *B-I* and 110 under *B-II*. The three tests, under *A*, *B-I*, *B-II*, may be conveniently applied simultaneously, as exhibited below for $r=16$, $t_1=5$. (Solutions of the ENSC type are ignored. Obviously unnecessary entries are omitted from the tabulation.)

i	t_i	*A* $2t_j$	*B-I* $2t_j-t_1$	*B-II* $r+2t_j-t_1$
1	5	10	5*	
2	10			
3	15			
4	4	8	3	
5	9	18	13	
6	14			
7	3	6	1	
8	8	16	11	
9	13			
10	2	4	−1	
11	7	14	9	
12	12			
13	1	2	−3	13*
14	6	12*	7	
15	11			

The asterisks indicate the numbers which pass the indicated test as the middle term of the inequality. Thus, for case *A*, $j=14$, giving solutions

$(32q+22,\ 16q+21,\ 16q+5)$ for $q=1,\ 2,\ \cdots$. For case B-I, $j=1$, giving solutions $(48q+20,\ 32q+26,\ 16q+5)$ for $q=2,\ 4,\ 6,\ \cdots$. For case B-II, $j=13$, giving solutions $(48q+28,\ 32q+26,\ 16q+5)$ for $q=1,\ 3,\ 5,\ \cdots$.

The three tests together are, from the nature of their construction, exhaustive, yielding, if we include the ENSC type solutions discussed under case B, all "primitive" solutions for a given choice of r and t_1. Non-primitive solutions are obtained by adding $2kc$ to a and to b.

Editorial Note. The editors wish to thank H. D. Grossman for assistance in preparing this solution for publication.

Self-reciprocal Curves

E 666 [1945, 218]. *Proposed by N. D. Lane, St. Andrew's College, Aurora, Ont.*

For what value of k is the curve $x^m y^n = k$ self-reciprocal with respect to the circle $x^2+y^2=1$?

Solution by Francis Hall, Hunter College. Let (x_1, y_1) be any point on the curve $x^m y^n = k$. The equation of the tangent to the curve at this point is

$$my_1x + nx_1y = (m+n)x_1y_1. \tag{1}$$

Let P_2: (x_2, y_2) be the pole of (1) with respect to the circle $x^2+y^2=1$. Then the polar of P_2 with respect to the circle, namely $x_2x+y_2y=1$, must be the same line as (1). Therefore

$$my_1/x_2 = nx_1/y_2 = (m+n)x_1y_1,$$

or

$$x_1x_2 = m/(m+n), \qquad y_1y_2 = n/(m+n).$$

Using the fact that (x_1, y_1) and (x_2, y_2) lie on the curve we then find

$$k = \pm\,[m^m n^n (m+n)^{m+n}]^{1/2}.$$

Editorial Note. Using homogeneous coordinates it is easy to show that the curve $f(x, y, z)=0$ will be self-reciprocal to the circle $x^2+y^2=z^2$ if and only if $f(x, y, z)$ divides $f(\partial f/\partial x, \partial f/\partial y, -\partial f/\partial z)$. Imposing this condition on the curve $x^m y^n = kz^{m+n}$ we readily find the values of k as given above.

A Finite Series

E 670 [1945, 218]. *Proposed by C. D. Olds, Purdue University*

Sum the series

$$\sum_{r=1}^{2n-1} (-1)^{r-1} r \Big/ \binom{2n}{r}.$$

I. *Solution by Irving Kaplansky, Columbia University.* Let S be the desired sum. Multiply the identity

$$\frac{2n+2}{2n+1}\,\frac{1}{\binom{2n}{r}} = \frac{1}{\binom{2n+1}{r}} + \frac{1}{\binom{2n+1}{r+1}}$$

by $(-1)^{r-1}r$ and sum from $r=1$ to $r=2n$. On the left side we obtain

$$(S-2n)(2n+2)/(2n+1),$$

and on the right

$$\frac{1}{\binom{2n+1}{1}} - \frac{1}{\binom{2n+1}{2}} + \cdots + \frac{1}{\binom{2n+1}{2n-1}} - \frac{1}{\binom{2n+1}{2n}} - 2n = -2n,$$

the binomial coefficients cancelling in pairs. Solving for S we find

$$S = n/(n+1).$$

II. *Solution by J. S. Frame, Michigan State College.* Let the desired sum be denoted by S, and define the related sum S':

$$S' = \sum_{r=1}^{2n-1} (-1)^{r-1} \Big/ \binom{2n}{r}.$$

Now we have

$$(r+1)\Big/\binom{2n}{r} = (2n-r)\Big/\binom{2n}{2n-1-r},$$

and consequently

$$S + S' - 1 \equiv \sum_{r=0}^{2n-1} (-1)^{r-1}(r+1)\Big/\binom{2n}{r} = 0, \tag{1}$$

since a reversal of the order of terms in the sum replaces each term by its negative. For the same reason

$$nS' - S \equiv \sum_{r=1}^{2n-1} (-1)^{r-1}(n-r)\Big/\binom{2n}{r} = 0. \tag{2}$$

Solving the two linear equations (1) and (2) simultaneously, we find

$$S = n/(n+1), \qquad S' = 1/(n+1).$$

Also solved by Murray Barbour, Scott Crom, and E. P. Starke.

A Hobo in Distress

E 671 [1945, 274]. *Proposed by Alice Malice, Owen Sound, Ontario*

A hobo has crossed three-eighths of a high railway bridge when he hears behind him an express train coming at sixty miles per hour. He can just save his

life by running to either end of the bridge. How fast can he run? (There is no question of acceleration or deceleration; the engineer is merciless and the man gets going at once.)

I. *Solution by C. W. Foard, Eastman Kodak Company, Rochester, N. Y.* The time required for the hobo to run 5/8–3/8, or 1/4, the length of the bridge is the same as for the train to cross the entire span. Therefore the hobo's speed is 15 m.p.h.

Note that the hobo's speed is independent of the length of the bridge. If the bridge is 480 feet long, the hobo is a grade school 100 yard dash man, requiring 13.6 seconds for the run. However, if the bridge is 1.6 miles long, his best effort is indeed noteworthy—the long awaited four minute mile! Shades of Gunder Haegg!

II. *Solution by W. C. Rufus, University of Michigan.* The time on the bridge in either direction the hobo ran would be proportional to the distance he ran and inversely proportional to his speed relative to the train. Let his top speed be x m.p.h. Then $3/(60-x)=5/(60+x)$, or $x=15$ m.p.h.

Also solved by E. F. Allen, D. W. Alling, N. H. Anning, Gene Archer, LeRoy Babcock, W. G. Brady, William Bunyan, H. N. Carleton, M. L. Constable, J. H. Cross, Monte Dernham, William Douglas, R. L. Duncan, Daniel Finkel, H. G. Funkhouser, C. B. Gass, E. B. Hancock, Frank Hawthorne, Robert Hoskins, O. J. Karst, Jr., Jerry Norris, E. K. Paxton, J. Price, A. R. Rhodes, E. D. Schell, M. J. Sheehy, E. S. Smith, F. C. Smith, Margaret Sproul, E. P. Starke, J. A. Tierney, W. Unterberg, Jeanette Van Os, R. H. Wilson, Jr., and an unsigned solver. N. H. Anning remarks that essentially this problem can be found on page 329 of Part II of the High School Algebra by Birchard and Robertson, Toronto, William Briggs, 1889.

Some Special Regular Polygons

E 673 [1945, 274]. *Proposed by L. S. Shively, Ball State Teachers College*

Does there exist a regular polygon having both these properties: (a) a diagonal is equal to the sum of two other diagonals; (b) a diagonal is equal to the sum of a side and another diagonal?

Solution by E. P. Starke, Rutgers University. By considering the angles at the center of a regular polygon of n sides and radius r, we have

$$d_j = 2r \sin j\pi/n, \qquad j = 1, 2, 3, \cdots, [n/2], \tag{1}$$

where d_1 is the length of a side and d_j is the length of a diagonal which subtends j sides. We require solutions of

$$d_m = d_p + d_q, \qquad 0 < p \leqq q < m \leqq \tfrac{1}{2}n.$$

This is condition (b) when $p=1$ and condition (a) when $p \geqq 2$. By (1) we have

$$\begin{aligned} 2 \sin (m\pi/n) &= 2 \sin (p\pi/n) + 2 \sin (q\pi/n) \\ &= 4 \sin \{(p+q)\pi/2n\} \cos \{(q-p)\pi/2n\}. \end{aligned}$$

An obvious solution is obtained by putting $(p+q)\pi/2n=\pi/6$ and $(q-p)\pi/2n=\pi/2-m\pi/n$. Then

$$q = n/3 - p, \qquad m = n/3 + p.$$

Evidently n must be a multiple of 3. For conditions (a) and (b) to hold simultaneously we need at least $p=1$ and $q=2$. Thus the conditions on p, q, m, and n require that $n \geqq 12$. Finally for any regular polygon of $3k$ sides ($k \geqq 4$), both conditions are satisfied. In particular

$$\text{if} \quad n = 12, \quad d_5 = d_1 + d_3 \quad \text{and} \quad d_6 = d_2 + d_2;$$

$$\text{if} \quad n = 15, \quad d_6 = d_1 + d_4 \quad \text{and} \quad d_7 = d_2 + d_3;$$

$$\text{if} \quad n = 18, \quad d_7 = d_1 + d_5 \quad \text{and} \quad d_8 = d_2 + d_4 \quad \text{and also} \quad d_9 = d_3 + d_3.$$

Also solved as above by J. B. Kelly and by the Proposer, who showed geometrically that the regular 18-gon possesses the required properties.

A Projective Invariant

E 675 [1945, 274]. *Proposed by Howard Eves, College of Puget Sound*

Show that the ratio of the curvatures of two curves in a plane at a point of contact is invariant under projection.

Solution by the Proposer. A. *Synthetic treatment.* Let the two curves have at P the common tangent PT. Let O be any fixed point not on PT and let a line through O cut the curves near P in Q and R. Let (Q) and (R) be the circles tangent to the curves at P and passing through Q and R respectively. Let the line through O cut (Q) again in q and (R) again in r. Then, as $T \to P$,

$$\begin{aligned} \lim (OQTR) &= \lim (OQ/OR)(TR/TQ) \\ &= \lim (OQ/OR)(Tq/Tr) \\ &= \lim (OQ/OR) \lim (Tq/Tr) = a/b, \end{aligned}$$

where a and b are the radii of curvature at P of the two given curves. (For circles (Q) and (R) approach the circles of curvature at P as limiting positions, and $Tq/Tr \to a/b$.)

Using bars in the projected figure we similarly find that as $\overline{T} \to \overline{P}$, $\lim (\overline{QQTR}) = \bar{a}/\bar{b}$. But $(OQTR) = (\overline{OQTR})$, whence $a/b = \bar{a}/\bar{b}$, and the theorem is proved.

B. *Analytic treatment.* Cartesian coordinate systems can be chosen on two planes p and $\bar{p}$ such that any given central projection of p on $\bar{p}$ takes the form

$$\bar{x} = mx/y, \qquad \bar{y} = mn/y.$$

Letting primes indicate differentiation with respect to y and $\bar{y}$ we then find

$$\bar{x}'' = m^2 n x''/\bar{y}^3.$$

Now let $x=f(y)$ and $x=g(y)$ be two curves in plane p, tangent to each other at

the point $P:(x_0, y_0)$, and let R be the ratio of the two curvatures at P. Then

$$R = \frac{-f''(y_0)}{(1+[f'(y_0)]^2)^{3/2}} \frac{(1+[g'(y_0)]^2)^{3/2}}{-g''(y_0)} = f''(y_0)/g''(y_0).$$

Similarly, the ratio of the curvatures of the two projected curves in plane $\bar{p}$ is

$$\begin{aligned}\overline{R} &= \bar{f}''(\bar{y}_0)/\bar{g}''(\bar{y}_0) \\ &= (m^2 n f''(y_0)/\bar{y}_0^3)/(m^2 n g''(y_0)/\bar{y}_0^3) \\ &= R,\end{aligned}$$

and the theorem is proved.

A. W. Tucker furnished for this theorem the following references from *Introduction à la Géométrie Projective Differentielle*, by Fubini and Čech (Paris, 1931), p. 18: "Ce rapport [des courbures euclidiennes des courbes C et Γ dans le point de contact] est donc (Smith [1] et Mehmke [1], [2]) un invariant projectif."

H. J. S. Smith, [1], London Math. Soc. 2, 1869, pp. 196–248.

R. Mehmke, [1], Zeit. für Math. 36, 1891, pp. 56–60, [2], Zeit. für Math. 36, 1891, pp. 206–213.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis 5, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4185. *Proposed by B. M. Stewart, Michigan State College*

In an unexplored region known as Wild Basin, a hunter found himself lost. But he had in hand a compass, and there were visible on two distant peaks fire ranger stations, A and B, whose bearings from his own cabin, O, he knew. From one observation point, C, the hunter took bearings on A and B; walking to another observation point, D, nearby, he took bearings on A, B, and C. Somehow he felt these seven bearings ought to enable him to find the direction homeward, *i.e.*, the bearing from D to O.

Show that the hunter's problem may be solved if he has either (1) mathematical tables or (2) a straightedge, in this case using the compass card as a protractor.

Note. Suggested by a correspondent of M. H. Ingraham, University of Wisconsin.

4186. *Proposed by Fritz Herzog, Michigan State College*

It is known from the theory of probability that for x fixed, $0<x<1$, the largest of the $n+1$ terms ${}_nC_r x^r(1-x)^{n-r}$, $r=0, 1, \cdots, n$, is asymptotically equal to $[2\pi nx(1-x)]^{-1/2}$, as $n\to\infty$. Show that for all integral values of n and r with $n\geqq 1$, $0\leqq r\leqq n$ and for all values of x with $0<x<1$

$$ {}_nC_r x^r(1-x)^{n-r} < 1/[2enx(1-x)]^{1/2} $$

and that this represents the best inequality in the sense that the numerator on its right cannot be replaced by any number less than unity.

4187. *Proposed by Victor Thébault, Tennie, Sarthe, France*

For a given tetrahedron the ratio of the distances of the Monge point and of the circumcenter to the common perpendicular to two opposite edges is equal to $\cos\theta$, where θ is the angle between the two edges.

SOLUTIONS

A Multiplication Puzzle

4135 [1944, 475]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

With the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, taken one at a time, form two numbers of five digits each such that one is a multiple of the other.

Solution by E. P. Starke, Rutgers University. Let N and aN be the two five-digit numbers. It appears there are 94 solutions corresponding to the following values of a and N:

$a=2$,	
1′35′48, 1′38′45, 1′3′485,	18 solutions
3′27′09, 3′29′07, 2′3′079,	12
1′4685, 1′4865, 15′486, 16′485, 45′186, 46′185,	12
2′0679, 2′0769, 27′069, 29′067, 267′09, 269′07;	6
$a=3$,	
16794, 17694, 20583, 23058, 30582, 32058;	6
$a=4$,	
15237, 17235, 17352, 21735, 23517, 23715, 20394, 20439;	8
$a=5$,	
13458, 13584, 13854, 14538, 15384, 18534, 14586, 14658,	
15846, 15864, 18564, 18654;	12
$a=6$	none
$a=7$,	
14076;	1
$a=8$,	
10459, 10469, 10537, 10579, 10592, 10674, 10679, 10742,	
10794, 10932, 10942, 10953, 10954, 12073, 12307, 12345;	16
$a=9$	
10638, 10647, 10836.	3

The values of N for $a=2$ are separated by the mark (′) into parts which can be rearranged without destroying the desired property. Thus 1′35′48 represents the six numbers 13548, 14835, 35148, 35481, 48135, 48351. Omitting permutations which introduce an initial zero, we have the number of solutions indicated. That there can be no value of N for $a=2$ without such a separation is evident from the following considerations: in N the only digits less than 5 are the initial digit and those which follow the mark (′); in aN the only even digits are the final digit and those which precede the mark; without a separation, N would require four digits greater than 4, while aN would require four odd digits, which is not possible.

For $a=5$, N cannot be odd, which may be seen as follows: since only the initial digit of N is 1, the digit 0 evidently cannot appear in $5N$; thus 0 is a digit of N; then the digit in the preceding place in $5N$ is either 0 or 5; since the final digit of $5N$ is 5 there is a duplication of either 0 or 5, contrary to hypothesis. Now when N is even, $\frac{1}{2}N$ is the same as $5N$ except that the digit 0 is placed at the other end. Hence all values of N for $a=5$ may be obtained from pairs N_1, $2N_1=N$, where N_1 has the initial digit 0. Such values of N_1 are obtained directly from the list above for $a=2$.

In general we have

$$10234 \leqq N \leqq 98765/a.$$

Also, since the sum of the digits of N and aN is 45, we have

$$(a+1)N \equiv 0 \bmod 9,$$

whence $N\equiv 0 \pmod 9$ for $a=3, 4, 6, 7, 9$, and $N\equiv 0 \pmod 3$ for $a=2, 5$. With these restrictions in mind it is easy to systematize the testing of possible values of N so as to detect duplication of digits in N and aN without unreasonable expenditure of time.

Solved also by Murray Barbour, Shepard Bartnoff, Monte Dernham, Irving Kaplansky, E. D. Schell, and the proposer.

Divisors of Factorials

4137 [1944, 533]. *Proposed by P. Erdös, Purdue University*

Given an integer $x \leqq n^2/4$ which has no prime factor greater than n, show that $n! \equiv 0 \bmod x$.

Solution by Irving Kaplansky, Columbia University. I shall establish the following generalization. Let $x \leqq n^k/k^k$, where k is a positive integer $\geqq 2$, and suppose that for any prime factor p of x, $p^{k-1} \leqq n$. Then $n! \equiv 0 \pmod x$. The proposed problem is the case $k=2$.

Proof. Clearly it suffices to consider the case where x is a prime power p^r with $r \geqq k$. Then $n! \equiv 0 \pmod{p^r}$ will follow a fortiori from $rp \leqq n$, or $r^k p^k \leqq n^k$. Since by hypothesis $p^r k^k \leqq n^k$, we need only prove

$$(1') \qquad r^k p^k \leqq p^r k^k$$

or

$$(1) \qquad f(r) = \frac{\log r - \log k}{r - k} \leqq \frac{\log p}{k}.$$

The numerator of $f'(r)$ is

$$g(r) = (r - k)/r - (\log r - k \log k).$$

Now $g'(r) = k/r^2 - 1/r \leqq 0$ for $r \geqq k$. Hence $g(r)$ is decreasing and, since $g(k) = 0$, $g(r) \leqq 0$ for $r \geqq k$, and hence finally $f(r)$ is a decreasing function of r for $r \geqq k$.

Case I. $p > 2$. Then $f(k) = 1/k < (\log p)/k$, and (1) is true for $r = k$, hence for all $r \geqq k$.

Case II. $p = 2$. Here (1) is an equality when $r = 2k$. Since (1') is an equality for $r = k$, we need only consider the case $k < r < 2k$.

Case II*a*. $k = 2$. Then $r = 3$, $x = p^r = 8$, $n \geqq \sqrt{4x} > 4$ and 4! is divisible by 8.

Case II*b*. $k = 3$. Then $r = 4,\ 5$; $x = 16,\ 32$; $n \geqq 3x^{1/3} \geqq 6,\ 8$. In each case x divides $n!$

Case II*c*1. $k \geqq 4$, $r \leqq 3k/2$. We are trying to prove that 2^r divides $n!$ Now by the argument that led to (1'), we know that $2k \leqq n$. Hence it will suffice to prove that 2^r divides $(2k)!$. The power of 2 which occurs in $(2k)!$ is

$$k + [k/2] + [k/4] + \cdots \geqq 3k/2 \geqq r.$$

Case II*c*2. $k \geqq 4$, $r > 3k/2$. From $x = 2^r \leqq n^k/k^k$, we have $n \geqq k\ 2^{r/k} > 4 \cdot 2^{3/2} > 11$. Now the power of 2 dividing $n!$ is

$$[n/2] + [n/4] + \cdots \geqq 3n/4 - 2,$$

and for $n \geqq 12$, $3n/4 - 2 > 3n/\sqrt{32}$. Hence to prove that 2^r divides $n!$ it will suffice to prove $r < 3n/\sqrt{32}$. Now repetition of the arugment which led to (1) yields (2) as the inequality which we wish to prove:

$$(2) \qquad F(r) = k(\log r - \log k) - r \log 2 + k \log (\sqrt{32}/3) \leqq 0.$$

$F(3k/2) = 0$, and $F'(r) = k/r - \log 2 < 2/3 - \log 2 < 0$, so that $F(r) \leqq 0$, as desired.

Solved also by D. H. Browne, J. B. Kelly and Sol Rubinow, and W. J. Robinson.

Editorial Note. The solution by Kelly and Rubinow may be put in the following form: If p is a prime factor of x, denoted by $p \mid x$; then, since $p \leqq n$, $p \mid n!$. Suppose first that the multiplicity of p in x is $2r$, $r \geqq 1$; then $2p^r \leqq n$, and it follows that $p^{2r} \mid n!$.

Suppose next that the multiplicity of p in x is $2s+1$. $s \geqq 1$, then $n \geqq 2\sqrt{p}\, p^s$. Consider *Case A* where $4p^s < n$. Here it easily follows that $p^{2s+1} \mid n!$. *Case B*, $4p^s \geqq n$. Here $4p^s \geqq n \geqq 2\sqrt{p}\, p^s$, $2 \geqq \sqrt{p}$, $2\sqrt{p} \geqq p$, $n \geqq p^{s+1}$. It now follows that $p^{2s+1} \mid n!$. This completes the proof that $x \mid n!$.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, N. Y.

Professor J. I. Tracey, Branch Head of the Mathematics Department of the Biarritz American University, announces the following staff: Sgt. E. F. Arndt, Frank Ayres, F. R. Bamforth, I. A. Barnett, Lt. Col. A. C. Cohen, Pfc. S. D. Conte, Cpl. J. C. Gibson, L. S. Hill, Capt. A. O. Lindstrum, R. B. McClenon, E. R. C. Miles, J. H. Neelley, A. L. Nelson, Lt. Stephen Oselinsky, C. G. Phipps, J. C. Polley, B. P. Reinsch, Capt. W. T. Scott, C. R. Sherer, Gabor Szegö, Lt. H. E. Temmer, H. S. Thurston, J. I. Tracey, Lt. Col. A. S. Turner, Major V. R. Walker, R. M. Winger.

Dr. E. E. Blanche, Branch Head of the Mathematics Department of the Florence American University, announces the following staff: Cpl. Thomas Anderson, T. A. Bancroft, H. C. Christofferson, W. H. Fagerstrom, C. G. Jaeger, T/Sgt. William Lassow, Lt. T. A. Love, Sk3C Emanuel Mehr (U.S.N.R.), Sgt. W. F. Prahl, Jr., Capt. M. C. Resick, Cpl. W. C. Stone.

Captain Aaron Bakst has been awarded the Bronze Star Medal for meritorious achievement in connection with military operations in the Southwest Pacific Area.

The University of Michigan announces: Professor C. E. Love is on leave of absence for the year 1945–46; Dr. R. C. F. Bartels has been promoted to an assistant professorship and, together with Assistant Professors G. E. Hay and Wilfred Kaplan, has returned to the University.

F. L. Celauro has been appointed to an assistant professorship at Lehigh University.

Assistant Professor Max Coral of Wayne University has been promoted to an associate professorship.

Paul Cramer has been appointed head of the physics department of Huron College, South Dakota.

Dr. F. C. Gentry has been appointed to an associate professorship at Louisiana Polytechnic Institute.

Dr. R. A. Good has been appointed to an assistant professorship at the University of Maryland.

Associate Professor E. H. Hadlock of Hastings College, Nebraska, has been promoted to a professorship.

Professor W. L. Johnson of Mississippi Southern College, Hattiesburg, has been appointed dean of men.

Dr. A. M. Peiser has been appointed head of the Statistics Research Group at the Langley Field laboratory of the National Advisory Committee for Aeronautics.

Dr. C. J. Thorne has been appointed to an assistant professorship at the University of Utah.

Lt. Col. W. H. White has resumed his duties as assistant professor of mathematics at Virginia Polytechnic Institute.

The following appointments to instructorships are announced:

Bergen Junior College, Teaneck, New Jersey: Edward Rayher

University of Kansas: Edison Greer

Vanderbilt University: J. A. Tierney

Professorship Emeritus M. J. Babb of the University of Pennsylvania died October 27, 1945.

Professor J. D. Tamarkin of Brown University died November 18, 1945.

GENERAL INFORMATION

EDITED BY C. V. NEWSOM

Send information of especial interest to mathematicians, exclusive of personal items, to C. V. Newsom, Oberlin College, Oberlin, Ohio.

UNIVERSAL MILITARY SERVICE IN PEACE TIME

A REPORT OF A SUBCOMMITTEE OF THE WAR POLICY COMMITTEE*

Foreword (prepared by Professor M. H. Stone, Chairman of the War Policy Committee): The War Policy Committee of the American Mathematical Society and the Mathematical Association of America was formed to study the many questions of professional and scientific policy arising out of the war. No subject has been of greater interest or more vital concern to the Committee than the relations between scientific effectiveness on the one hand and the military requirements of the nation on the other. A most important aspect of this subject is treated in the report on Universal Military Service in Peace Time

* Prepared in July, 1945, by a subcommittee consisting of Professors W. L. Hart (Chairman) Saunders Mac Lane, and C. B. Morrey, Jr., and approved by the War Policy Committee. This report has also been approved by the Council of the American Mathematical Society and the Board of Governors of the Mathematical Association of America.

which is now made public. This report, prepared some time ago by a special subcommittee, is directed in the main at points upon which mathematicians as such are particularly qualified to express informed opinions. Whatever view may ultimately prevail concerning universal military training in peace time—and it should be emphasized that there are many citizens, mathematicians included, who doubt the wisdom of introducing such a peace-time military program—it is clearly of the first importance that no program deleterious to the scientific and technological vigor of the nation should be adopted. The report deals frankly and in detail with this vital segment of the problem now before Congress. In offering the recommendations of the report as a professional contribution to the current discussion, the War Policy Committee hopes to render a modest public service strictly within the natural sphere of its activity.

1. *Introduction.* In the Congress of the United States, bills are under consideration (S. 188 and H.R. 515, companion bills) which would prescribe a year of military service in peace time for all young men of suitable physical qualifications. These bills present the nation with a proposal for a fundamental new departure in our national life and it is appropriate that the widest discussion should occur concerning all sides of the situation which would result. It is natural that the greater part of the arguments pro or con about S. 188 in the public press should deal with military, internal political, and international aspects of the matter. Various spokesmen of the general field of education, particularly at the college level, as well as of specific fields of learning have likewise discussed the measure under consideration. It is the main purpose of this report to state opinions concerning the impact of compulsory military service in peace time on those phases of our national life which are of primary significance to mathematicians and which lie in areas where we may speak with special authority. In carrying out this aim, it is natural that at times we shall phrase remarks from a general educational viewpoint not primarily associated with the field of mathematics. A summary of the opinions advanced in the report is to be found in Section 11.

2. *Orientation of the report.* In the discussion of any measure like S. 188, we may conceive of a division of interest between two major questions. First, one may ask, "Should the United States have universal military service in peace time?" Second, "If Congress is to pass such an act, what provisions should it contain in order that the greatest possible good should be obtained for the nation as a whole and for the young men who will perform the service?" We realize that these questions are considerably interrelated but, for our purposes, we shall act as if they are relatively independent. It is probable that the decision on the first question should and will be reached mainly on the basis of significant national and international considerations of a social, economic, political, and military nature. In this connection it is likely that only small weight will be given to those points where our opinions as mathematicians or educators would be of any special importance as compared to those ideas we may express individually as mere citizens. Hence we have decided that, in regard to the first

question and the most intimately connected phases of the second question, we shall make only a single statement.

> We believe that a decision about introducing universal military service in peace time should be reached only after an investigation by a carefully selected commission, preferably appointed by the President of the United States. We recommend that this commission should include not only representatives of the armed forces but also individuals from all other important sections of our population, with a strong representation from the fields of industry, science, and technology which have been of such great importance during the present war. This commission should have the wide objective of considering what combination of universal military service, scientific development work on weapons of war, and associated industrial and educational measures would be best adapted to insure the safety of the nation.

The balance of this report will be concerned with matters particularly germane to a discussion of the second of the two major questions which we have mentioned, although we have no intention of considering all its aspects. Our remarks will be phrased under the explicit assumption that *universal military service in peace time will be adopted*. However, this viewpoint is not to be taken as evidence that we desire or expect a bill such as S. 188 to become law.

For future reference, we note that the present bill S. 188 includes the following features.

> 2.1. The bill omits specification of all subsidiary values as justification for military training and is based only on reasons relating to national security and the prevention of dissipation of training resources and experience.
> 2.2. The bill contains provision for a period of four years (ages 18 to 22) in which the individual may select the year for military training. A high school graduate with his parents' consent may volunteer to start his year of training while he is still 17 years of age.
> 2.3. The year of training will be used by the Army and Navy as a basis for selecting non-commissioned and commissioned officers.

3. *Background relating to the present war*. The importance of technical developments of new weapons and tactics during the military operations of the last few years has emphatically shown that science has made a fundamental contribution to the military power of the nation. More particularly, the abstract science of mathematics has been put to effective application in a surprising number of fields, including even the extensive use of mathematicians in operational analysis work overseas. These facts are important in the formulation of an intelligent long term policy for maintaining the nation's military strength. It can safely be said that the military potential of a country does not depend solely upon the size and character of its armed forces. This potential also depends largely upon the technical and industrial talent available and upon the vigor, ability, and training of its scientific and technical experts.

> Long range military planning, therefore, requires that able young men be encouraged both in preparing themselves in basic scientific and mathematical knowledge and in acquainting themselves with the ways in which this knowledge can be put to use in time of war or national emergency.

With the preceding viewpoint in mind, we deeply regret to note that the manpower policies of the United States during the war have not been well formulated with respect to the utilization of men trained in science and technology or potentially able to assimilate such training. Draft deferments for the purpose of obtaining certain varieties of technical training have been at times difficult and often impossible to obtain. Some valuable technical men have been wasted in non-technical activities and others would have been wasted had it not been for the energetic effort of representatives of science and technology in prodding reluctant or unenthusiastic government bureaus into action. During the war the United States has followed a sadly shortsighted and unintelligent policy with regard to science and technology by not providing for appropriate annual increments of undergraduate men to be trained in those fields. There are facts at hand which indicate that England and probably Russia have not made a similar error during the war. We agree strongly with the many opinions which have recently been expressed concerning the necessity for prompt resumption of training for the scientific professions, even before the end of the war with Japan. We believe that any additional delay in provisions for such training may have fatal effects not only on national health and the technological side of our economy but, also, on our means for future national defense. These remarks are well justified by the established facts about the present and potential shortages of physicians, dentists, Ph.D.'s in the various fields of the physical sciences, and technologists. This background leads us to the following conclusion.

> We consider it very important that universal military service in peace time should not be planned in such a way as to interfere with efforts, first, to eliminate as quickly as possible the present shortage of scientists and technologists and, second, to provide for a continuous generous supply of such essential categories of trained citizens in the future.

4. *An attitude about exemptions from military service.* It has been the traditional American viewpoint that all men should be treated as nearly alike as possible in fundamental respects whenever a call for universal military service is issued by our Government. It could be argued that this was a major cause for some of the short-sighted manpower policies of the present war period which have been referred to previously. However, we believe that it would have been possible for Selective Service to have been administered during the war without violating the specified tradition and, also, without introducing various major evils. This democratic viewpoint is so deeply seated that, regardless of whatever arguments might be presented against it, neither the general public nor the affected young men could be expected to give friendly support to measures for creating numerous exempt classes of the population in case universal military

service in peace time should be adopted. Realistically, we appreciate that any request for the exemption of mathematicians or physical scientists, for instance, could be expected to induce requests for the exemption of social scientists, and scholars in almost all other fields of learning, as well as representatives designated by innumerable pressure groups outside the field of education. The result of such requests for exemptions would undoubtedly be the creation of an ironclad policy of *no* exemptions. The requests might even develop an unsympathetic attitude among the directors of military service with respect to rational plans for differentiation of training, which we propose to emphasize in place of exemptions.

We shall proceed, then, with the explicit premise that it would be undesirable to suggest the exemption of men with special mathematical talent and that each man, regardless of his abilities, will have to spend one year of his life in military service.

5. *A liberal interpretation of the words "military service."* The present rate of increase in the military uses of science and technology convinces one that success in any future war will be dependent on continued progress in science and on the existence of a large reservoir of trained technologists. Thus it is essential that any program for universal military service should be consistent with associated plans for the development of our scientific and technological potential. It follows that the final program should not be organized with the narrow objectives of merely disciplining masses of men and preparing them for handling existing weapons of war. Hence, the term "military service" should be interpreted in a liberal fashion.

"Military service" should include not only the usual routine service in the Armed Forces but, also, various other highly technical forms of training which are just as essential for enhancing the military strength of the nation. These types of service should be made available for properly qualified young men under some studied system for differentiation of training in accordance with ability. In particular, the system should provide appropriate advanced types of service in the case of the small but important group having special aptitude for and training in mathematics, the physical sciences, or technology.

6. *Candidates for the most highly technical military service.* During the present war, under Selective Service the Armed Forces have made intelligent efforts to place the inductees into various enlisted classifications in accordance with the abilities shown by the men in civilian life and in various tests given by the Armed Forces. Also, similar efforts have been made to locate men with appropriate backgrounds as candidates for commissioned ranks in various branches of the Armed Forces. It should be anticipated that these commendable practices in the direction of differentiated training would continue under universal military service in peace time. The Committee then directs the main force of its recommendation for differentiated training at the cases of the exceptionally

superior young men who, let us say, have the requisite mathematical and scientific aptitude and inclination to become Ph.D.'s in mathematics or a physical science, or to advance to the highest levels of attainment in an engineering field. Under the reasonable hypothesis that less than one-third of the men in our population with such potentialities finally pursue careers in the indicated fields, we are led to estimate the number of men of this sort available in any annual age group at about 3000 men. If we should attempt to sift them out of the enormous annual group of high school graduates by any existing devices for the prediction of mathematical or scientific aptitude, the percentage of error in diagnosis would be very high. It is likely that we would have to earmark about 20,000 boys annually in order to be reasonably certain that the final group would include 90% of the 3000 boys whom we have mentioned. Such an attempt at selection of 20,000 boys would entail severe difficulties in the field of educational measurements and also many later complications if the members of the earmarked group were to be given individual attention during the next four to seven years. Our final plan will indicate a procedure for obtaining such candidates which appears more simple than an attempt to earmark them at an early age. At present we offer merely the following conclusion. Among the boys who graduate from high school in any year there is a group of about 20,000 with such high mathematical and scientific aptitude that they deserve special consideration under any plan for universal military service. This group will hereafter be referred to as the *select technical group*.

7. *Advantages of deferment of military service for the select group.* It is likely that any boy in the group would graduate from high school by the time he is 18. Regardless of considerations related to military service, the rarity of the aptitudes present in the group makes it desirable that each member should be shielded from unnatural influences which might lead him away from the fields of science and technology where his abilities would be so valuable to the nation. We believe that he should be encouraged to carry out any latent inclination to enter upon scientific or technical training. Immediately after graduation from high school, an interruption of a full year in the educational process might eliminate or dim his embryonic urge to scientific study. The interruption would also cause an undesirable break in the important preliminary stages of progress in mathematics and science. Moreover, a year of military service at age 17 or 18 might expose the boy to certain military, social, or industrial activities which, although appealing to a youth, could lead him after a few years to an undesirably low terminal point outside of advanced science or technology, with a corresponding loss to the nation.

If our attention is now fastened primarily on differentiated training at a high level under military service, we can likewise argue that a delay beyond age 18 in this service may be desirable for a boy in the select group. When he graduates from high school, he will not yet possess the requisite mathematical and scientific background to be eligible for or to obtain maximum good from many of the high grade types of differentiated service which can be made available. Before

he receives college training, his possibilities might be merely those of an exceptionally intelligent young man with latent scientific abilities and interests but no particularly useful background knowledge. He might even gain a permanently wrong orientation with respect to various aspects of military service where he should eventually become most useful to the nation in case of war.

The preceding remarks lead us to advocate the following flexible provision, similar to Item 2.2, in any final regulations governing universal military service in peace time.

> The directors of the program should have the power to approve a request for deferment of the year of service for any young man when this is deemed best for the interest of the nation. In particular, continued efficient progress with high achievement in the study of mathematics, a physical science, or technology, even through the stage of a Ph.D. degree, should be considered sufficient cause for such deferment.

Conceivable complications in the administration of the rules for deferment should not be considered a deterrent to the adoption of the preceding recommendation. It should be possible to develop a smooth routine for such a process in peace time which would be superior to the method employed with respect to draft deferments of scientists and scientific students during the war. In this connection it is pertinent to observe that a proposal for special treatment immediately under Selective Service for a select group of 20,000 was presented to Congress recently.

8. *Suggestions concerning highly differentiated service.* We proceed under the assumption that the deferment recommendation of Section 7 could be adopted. Then, we propose the following rough outlines of a plan for implementing the suggestions about differentiated service.

8.1. A board should be created to canvass the possibilities for in service training on intermediate and higher technical levels for properly qualified applicants and to supervise plans for instituting such training. This board should involve civilian representatives of the fields of mathematics, the physical sciences, and engineering, together with representatives of the Armed Forces.

8.2. The search for training locations should cover the various laboratories, research divisions, arsenals, and other technical activities of the Army and Navy. Also, the search should extend into associated private industries and related curricula and research in universities. As a mere sample of possibilities we mention the following types of training which might be given to properly selected groups each year.

8.21. An advanced course in meteorology for the future use of the Army or Navy, along the lines of the curriculum presented at five major universities in the United States during the early years of the war, and associated field experience.

8.22. In service training at the Ballistic Research Laboratory of the

Army Ordnance Department at the Aberdeen Proving Ground.

8.23. In service training in the artillery fire control division at an ordnance plant.

8.24. In service training at a laboratory of the Signal Corps, or at an aeronautical research laboratory of either the Army, the Navy, private industry, or a university.

8.25. Study of the mathematical and operational phases of cryptanalysis in the intelligence division of the Army or Navy.

8.3. For each of the training possibilities, the Board should decide upon appropriate academic prerequisites to be demanded of any man who is to qualify for the opportunity. These prerequisites should probably be much more extensive than for any similar training during the war, because in peace time interested men would have ample opportunity to obtain the necessary background. A pamphlet should then be published listing the available varieties of technical training which might be taken as "military service" so that interested boys, even early in their high school life, could begin to orient their school work with respect to the future opportunities in military service. The pamphlet should urge boys of proper aptitudes to defer their entrance to military service until they gain the prerequisites for the type or types of service which appeal to them.

8.4. If administratively possible, assignments to any one of the most exacting and desirable varieties of training should be made on the basis of applications asserting definite interest in the field involved, as well as possession of the necessary prerequisites.

8.5. In addition to the technical training experience mentioned previously, an appropriate amount of routine "boot" training should be given either simultaneously with the technical work or in a separate period of time during the year of service.

8.6. The directing Board should examine the possibility of granting commissions in the Reserve Corps of the Army or Navy for satisfactory completion of certain types of technical training which would be available to the most highly qualified young men during their year of service. Acceptance by a young man of the opportunity of such training should not commit him to accepting such later commission as might be offered in this connection.

9. *Civilian educational by-products of universal military service.* If the interesting nature of the various technical types of differentiated military service, at low or intermediate as well as at high levels, and the corresponding academic prerequisites are properly advertised, strong repercussions might occur in the field of secondary education. It is probable that boys of ability and their parents would then demand maximum mathematical and scientific opportunities in the secondary curriculum for students of proper ability. Also, regardless of the possibility of differentiated training, it is our belief that a requirement of a year of military service, with the attendant delay in the study of the professions, would place an added demand for educational efficiency on both the high

schools and colleges. The knowledge resulting from a brief exposure to substantial secondary mathematics or science is easily forgotten. On the other hand, a long exposure to these fields gives the student a body of content which, even though it should remain dormant during a year of military service, would be effective in the next year after a brief review.

Hence, if universal military service is adopted, added efforts should be made in the secondary field to expose the better students to as much substantial mathematics and physical science as possible.

10. *Educational responsibilities of the Armed Forcs resulting from universal military service.* Even without the delay which would be caused by military service, the time for completing professional training encroaches on the age of greatest productivity and ingenuity as well as on the normal age for assumption of family responsibilities. Hence, under universal military service, the Armed Forces will be presented with a challenge to compensate as much as possible for the year of delay which would be introduced. It is obvious that the handicaps suffered by a young man due to an interruption of a year in his formal education would be considerably lessened if he should systematically carry on correspondence study during the year.

If universal military service is adopted, the Armed Forces should offer to the men in training the fullest cooperation and encouragement in the extension of their formal education by means of correspondence courses. At the college level, we recommend that this be done by paying the registration fees for men who take and diligently pursue courses offered by regular college correspondence departments, rather than through such an agency as the present Armed Forces Institute.

11. *Summary.* In the development of this report we have emphasized the following points and recommendations.

11.1 A joint civilian and military commission should be appointed by the President of the United States to study the problem of universal military service in peace time before final action is taken on the matter.

11.2. An expression of opinions by the American Mathematical Society and the Mathematical Association of America should be focused on constructive suggestions about the administration of universal military service if it should be adopted and about the solution of resulting educational problems.

11.3. No outright exemptions from universal military service should be requested.

11.4. The required military service should be highly differentiated in accordance with the aptitudes and training of the young men involved, with emphasis on exceptional differentiation for those few with the greatest technical abilities.

11.5. A system for deferring the year of military service should be instituted so that gifted young men might prepare themselves for advanced varieties of differentiated service before entering the Armed Forces.

11.6. The possibilities for technical varieties of differentiated service should be canvassed by a joint civilian and military board. The resulting training programs with the corresponding academic prerequisites should be well advertised among high school boys, their parents, and the teachers and administrators in the secondary field.

11.7. If universal military service is adopted, the field of secondary education will have the added responsibility of increasing the efficiency and quantity of instruction given in mathematics and physical science to the students of better than average ability.

11.8. The Armed Forces should encourage young men to continue their education during military service by taking correspondence work through regular school channels, and the Armed Forces should pay the costs of such study if a man carries it through diligently.

STUDENT DEFERMENT

On September 18, the National Headquarters of Selective Service issued the following regulation regarding the deferment of college students.

"Any person who entered upon a course of instruction at a college or university before he became eighteen years of age and who is ordered to report for induction during a quarter or semester of such course of instruction shall, upon his request, have his induction postponed (1) until the end of such quarter or semester, or (2) until he ceases to pursue continuously and satisfactorily such course of instruction, whichever is the earlier."

CALENDAR OF FUTURE MEETINGS

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., April 13, 1946.
ILLINOIS, Normal, May 10–11, 1946
INDIANA
IOWA, Grinnell, April 19–20, 1946
KANSAS
KENTUCKY, Louisville, April 27, 1946.
LOUISIANA-MISSISSIPPI, Ruston, La., March 15 and 16, 1946
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, New York, N. Y., May 11, 1946
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA
OHIO, April 4, 1946
OKLAHOMA
PACIFIC NORTHWEST
PHILADELPHIA, Philadelphia, November 30, 1946
ROCKY MOUNTAIN, Boulder Colo., April 19–20, 1946
SOUTHEASTERN, Raleigh, N. C., April 19–20, 1946
SOUTHERN CALIFORNIA, Pasadena, March 9, 1946
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May, 1946

CALCULUS

by Nelson, Folley, and Borgman

For the beginning student who needs the subject primarily as a tool in engineering and other scientific fields. Provides accurate statements of fundamental principles and theorems.

366 pages. $2.75.

D. C. HEATH AND COMPANY

Boston New York Chicago Atlanta San Francisco Dallas London

BOOK NEWS

Raymond W. Brink's

ANALYTIC GEOMETRY, Revised Edition

THIS text provides a rich, complete, and adaptable course in analytic geometry with an adequate introduction to solid analytic geometry. The emphasis throughout is upon method and logic making the text especially useful as preparation for calculus and other mathematical studies. Problems, distinctive for their variety and freshness, and carefully graded as to difficulty, progress from the purely formal requiring technical skill to those demanding originality and the power of analysis. *8vo, 350 pages.* $2.90

ESSENTIALS OF ANALYTIC GEOMETRY

THIS is a simpler, more flexible, and somewhat less detailed presentation of the material in *Analytic Geometry, Revised Edition.* It provides all of the essentials in method, information, and problems for preparation for calculus and for the development of general mathematical maturity and background. *Small 8vo, 233 pages.* $2.40

D. APPLETON-CENTURY COMPANY

35 West 32nd Street New York 1, New York

PRINCETON MATHEMATICAL SERIES

Edited by
MARSTON MORSE H. P. ROBERTSON A. W. TUCKER

1. **The Classical Groups.**
By HERMANN WEYL — Temporarily out of stock
2. **Topological Groups.**
By L. PONTRJAGIN (translated by E. Lehmer)
Temporarily out of stock
3. **An Introduction to Differential Geometry.**
By LUTHER PFAHLER EISENHART — 304 *pages,* **$3.50**
4. **Dimension Theory.**
By WITOLD HUREWICZ & HENRY WALLMAN — 165 *pages,* **$3.00**
5. **Analytical Foundations of Celestial Mechanics.**
By AUREL WINTNER — 448 *pages,* **$6.00**
6. **Laplace Transform.**
By DAVID VERNON WIDDER — 406 *pages,* **$6.00**
7. **Integration**
By E. J. MCSHANE — 400 *pages,* **$6.00**
8. **Theory of Lie Groups.**
By C. CHEVALLEY — In preparation

The prices above are subject to an educational discount of fifteen per cent.

PRINCETON UNIVERSITY PRESS
Princeton, New Jersey

THE CARUS MONOGRAPHS

No. 1. *Calculus of Variations,* by PROFESSOR G. A. BLISS. (First Impression, 1925; Second Impression, 1927; Third Impression, 1935; Fourth Impression, 1944.)

No. 2. *Analytic Functions of a Complex Variable,* by PROFESSOR D. R. CURTISS. (First Impression, 1926; Second Impression, 1930; Third Impression, 1943.)

No. 3. *Mathematical Statistics,* by PROFESSOR H. L. RIETZ. (First Impression, 1927; Second Impression, 1929; Third Impression, 1936; Fourth Impression, 1943.)

No. 4. *Projective Geometry,* by PROFESSOR J. W. YOUNG. (First Impression, 1930; Second Impression, 1938.)

No. 5. *History of Mathematics in America before 1900,* by PROFESSORS DAVID EUGENE SMITH and JEKUTHIEL GINSBURG. (First Impression, 1934.)

No. 6. *Fourier Series and Orthogonal Polynomials,* by PROFESSOR DUNHAM JACKSON. (First Impression, 1941.)

No. 7. *Vectors and Matrices,* by PROFESSOR C. C. MACDUFFEE. (First Impression 1943.)

Price $1.25 per copy to members of the Mathematical Association, one copy to each member, when ordered directly through the office of the Secretary, MCGRAW HALL, Cornell University, ITHACA, N.Y.

Additional copies for members, and copies for non-members, may be purchased from the Open Court Publishing Co., La Salle, Illinois, at the regular price of $2.00 per copy.

Join the National Council of Teachers of Mathematics

I. The National Council of Teachers of Mathematics carries on its work through two publications.

1. *The Mathematics Teacher.* Published monthly *except in June, July, August and September.* It is the only magazine in America dealing exclusively with the teaching of mathematics in elementary and secondary schools. Membership (for $2) entitles one to receive the magazine free.

2. *The National Council Yearbooks.* The first and second yearbooks are now out of print. The third on "Selected Topics in Teaching Mathematics," the fourth on "Significant Changes and Trends in the Teaching of Mathematics Throughout the World Since 1910," the fifth on "The Teaching of Geometry," the sixth on "Mathematics in Modern Life," the seventh on "The Teaching of Algebra," the eighth on "The Teaching of Mathematics in Secondary Schools," the ninth on "Relational and Functional Thinking in Mathematics," the eleventh on "The Place of Mathematics in Modern Education," the twelfth on "Approximate Computation," the thirteenth on "The Nature of Proof," the fourteenth on "The Training of Mathematics Teachers of Secondary Schools," the fifteenth on "The Place of Mathematics in Secondary Education," the sixteenth on "Arithmetic in General Education," —each may be obtained postpaid for $1.75; the seventeenth yearbook, "A Source Book of Mathematical Applications" and the eighteenth on "Multi-Sensory Aids in Teaching Mathematics" may be had for $2.00 each postpaid, from the Bureau of Publications, Teachers College, 525 West 120 Street, New York 27, New York. All of the yearbooks except the first two (3 to 18 inclusive) may be had for $22.00 postpaid.

II. The Editorial Committee of the above publications is W. D. Reeve of Teachers College, Columbia University, New York, Editor-in-Chief; Dr. Vera Sanford, of the State Normal School, Oneonta, N.Y.; and W. S. Schlauch of Hasbrouck Heights, N.J.

MEMBERSHIP BLANK

Fill out the membership blank below and send it with Two Dollars ($2.00) to THE MATHEMATICS TEACHER, 525 West 120th Street, New York 27, N.Y.

I, ... (LAST NAME) ..... (FIRST NAME) ....., *wish to become a member of the National Council of Teachers of Mathematics.*

Please send the magazine to:

..... (STREET AND ZONE NO.) ..... (CITY)

..... (STATE) ..... (WHEN TO BEGIN)

Please indicate here whether this is a new subscription or renewal .....

CALCULUS

By George E. F. Sherwood and Angus E. Taylor

One of the most comprehensive treatments of the basic principles, methods and uses of the calculus available. It abandons the sharp separation of differential and integral calculus in favor of a more natural approach emphasizing fundamentals. "Sets a new high standard for books for the first course in differential and integral calculus."—*Science.*

502 *pages, College List,* $3.75

★

WARTIME REFRESHER IN FUNDAMENTAL MATHEMATICS

By Lt. W. C. Eddy, Ens. E. S. Pulliam, A. H. Brolly, E. C. Upton and G. W. Thomas

256 *pages, College List,* $1.05

★

MATHEMATICS IN AGRICULTURE

By Roger V. McGee, Assistant Professor of Mathematics, Agriculture and Mechanical College of Texas

189 *pages, College List,* $2.50

★

VECTOR ANALYSIS

By James H. Taylor

The treatment in this text, while highly mathematical, includes many applications to the geometry of curves and surfaces, the motion of rigid bodies, and the flow of fluids. Because students are never permitted to lose sight of the practical functions of vector analysis, it is an excellent text for beginning engineering students. 180 *pages, College List,* $2.85

SEND FOR YOUR APPROVAL COPIES TODAY

PRENTICE-HALL, INC. 70 FIFTH AVENUE, NEW YORK 11

Distinctive McGraw-Hill Books

ANALYTIC GEOMETRY

By Ross R. Middlemiss, Washington University. 306 pages, $2.75

Designed especially to meet the needs of scientific and technical students. Covers conic sections, polynomial curves, and exponential, logarithmic, trigonometric, and inverse trigonometric functions and their graphs.

DIFFERENTIAL AND INTEGRAL CALCULUS

By Ross R. Middlemiss. 416 pages, $2.50

Intended for the usual beginning course, this book maintains a high degree of accuracy, and is at the same time exceptionally clear and readable. The book has been carefully planned to give the student a real understanding of the subject.

ALGEBRA: A SECOND COURSE

By R. Orin Cornett, Lecturer, Harvard University; on leave from Oklahoma Baptist University. 317 pages, $2.00

An entirely new approach, based on the idea that mastery of detail and routine operations should be preceded by an understanding of purpose, significance, and relation to the whole.

ANALYTIC GEOMETRY. *New third edition*

By Frederick S. Nowlan, University of British Columbia. In press—ready in January

Mathematically sound, easy to understand, and stimulating to the imagination, the new edition of this well known text contains an unusual number of distinctive features: the study of plane geometry is based upon the use of direction cosines; the study of conics is based upon the definition of *the general conic;* and polar coordinates are treated frm a new point of view.

THE DEVELOPMENT OF MATHEMATICS. *New second edition*

By E. T. Bell, California Institute of Technology. 618 pages, $5.00

As before, this widely used book tells the absorbing story of the part played by mathematics in the evolution of civilization, from about 4000 B.C. to the present day. The revision contains new material concerned with recent trends and developments in modern mathematics.

MATHEMATICAL THEORY OF ELASTICITY

By I. S. Sokolnikoff, University of Wisconsin. 373 pages, $4.50

Writing from the point of view of a mathematical physicist, the author provides a thorough foundation in the mathematical theory of elasticity, followed by the application of the theory to problems on extension, torsion, and flexure of isotropic cylindrical bodies.

Send for copies on approval

McGRAW-HILL BOOK COMPANY, Inc.

330 West 42nd Street **New York 18, N.Y.**

COLLEGE TEXTS

COMPUTATION AND TRIGONOMETRY

By Harold J. Gay. Beginning with the general angle, this book gives a more detailed discussion of computation than is ordinarily included in trigonometry texts. Starting with Chapter IV, the book presents the material generally covered in a text on plane and spherical trigonometry. With tables, $2.50. Without tables, $2.00

A SHORT COURSE IN TRIGONOMETRY

By James G. Hardy. This work differs significantly from the usual text in that numerical solution of triangles by logarithmic methods has been relegated from the customary first place in importance to a secondary position. 2nd Edition. With tables, $2.25. Without tables, $1.75

CALCULUS

By Robbins & Little. In this text topics usually presented in an introductory chapter are taken up throughout the book as the need arises and each new principle is presented by means of a specific problem. Special attention has been paid to the exercises of which there are nearly 2300. $3.25

A FIRST COURSE IN DIFFERENTIAL EQUATIONS

By Nelson B. Conkwright. Improvement in the exposition of tried material rather than the introduction of novel features has been the author's aim in writing this text. The material is developed with applied problems revealing the lines of investigation in which differential equations can be put to practical use. $2.25

SURVEY OF MODERN ALGEBRA

By Birkhoff & MacLane. Presupposing nothing but high school algebra, this text gives an integrated study of the theory of equations, elementary number theory, and modern algebra, with emphasis on the developments in modern algebra which have exerted influence on modern physics, philosophy, and higher mathematics. $3.75

THE MACMILLAN COMPANY
60 FIFTH AVENUE NEW YORK 11, N. Y.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

2

...NOLOGY DEPT.

THE AMERICAN MATHEMATICAL MONTHLY

DEVOTED TO THE INTERESTS OF
COLLEGIATE MATHEMATICS

...LIC LIBRARY
MAR 1 1946
DETROIT

VOLUME 53 NUMBER 2

CONTENTS

Distributions Derived from the Multinomial Expansion G. B. PRICE 59

The Fermat and Hessian Points of a Triangle . . . H. E. FETTIS 74

A Vector Interpretation of the Derivative Circle . . . . . H. R. PYLE AND BEVERLY M. BARKER 79

A Problem in Friction . . . . . F. A. VALENTINE 83

Discussions and Notes . . . . . J. H. M. WEDDERBURN, N. S. MENDELSOHN, VICTOR THÉBAULT 86

Clubs and Allied Activities 90

Recent Publications 93

Problems and Solutions 97

News and Notices 111

General Information 113

New Selective Service Policy Pertaining to Scientists 113

College Mathematics Credit and Vocational Advice for the Veteran . . . . . C. C. HURD 116

Calendar of Future Meetings 120

FEBRUARY 1946